

ON THE REPRESENTATION OF ORDER CONTINUOUS OPERATORS BY RANDOM MEASURES

BY

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ABSTRACT. Using the representation $Tf(y) = \int f d\nu_y$, where (ν_y) is a random measure, we characterize some interesting bands in the lattice of all order-continuous operators on a space of measurable functions. For example, an operator T is (lattice-)orthogonal to all integral operators (i.e. all ν_y are singular) or belongs to the band generated by all Riesz homomorphisms (i.e. all ν_y are atomic) if and only if T satisfies certain properties which are modeled after the Riesz homomorphism property [31] and continuity with respect to convergence in measure. On the other hand, for operators orthogonal to all Riesz homomorphisms (i.e. all ν_y are diffuse) we give characterizations analogous to the characterizations of Dunford and Pettis, and Buhvalov for integral operators. The latter results are related to Enflo operators, to a result of J. Bourgain on Dunford-Pettis operators and martingale representations of operators.

1. Introduction. In this paper we consider a kind of operator representation which generalizes the classical form of an integral operator

$$(1.1) \quad Tf(y) = \int k(y, x)f(x) d\mu(x),$$

or of a Riesz homomorphism

$$(1.2) \quad Tf(y) = a(y) \cdot f(\sigma(y))$$

by replacing the kernel function $y \rightarrow k(y, \cdot) \cdot \mu$ or the point-measures $y \rightarrow a(y)\delta_{\sigma(y)}$ by a more general stochastic kernel. Let E and F be Riesz function spaces over standard measure spaces (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) respectively (for notations and definitions see below). For every order continuous operator $T: E \rightarrow F$, there is a random measure $(\mu_y)_{y \in Y}$ of measures on (X, \mathcal{A}) such that for all $f \in E$

$$(1.3) \quad Tf(y) = \int f d\mu_y \quad \nu\text{-a.e.}$$

This kind of representation first appeared in probability theory: a regular conditional probability distribution is a representation for a conditional expectation operator. In the context of operator characterizations of Markovian kernels the

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special case $E = F = L_\infty$ is given in Neveu's book [36, §5.4]. The case $E = F = L_2$ was shown by Arveson [1] who used this representation in his study of operator algebras. For $E = F = L_1$, N. J. Kalton [19–21] gave an independent proof and applied it to the structure theory of L_p -spaces and Banach lattices (see also [6]). The general statement (1.3) is due to Sourour [41, 42].

Already in the theory of Markov chains (see [13, Chapter V]), but also in the above applications, it turned out to be useful to decompose T into its “integral” and its “singular” part:

$$(1.4) \quad Tf(y) = \int f d\nu_y^i + \int f d\nu_y^s, \quad \nu_y^i \ll \mu, \nu_y^s \perp \mu,$$

where $\nu_y = \nu_y^i + \nu_y^s$ is the Lebesgue decomposition with respect to μ , and also into its ‘diffuse’ and ‘atomic’ part:

$$(1.5) \quad Tf(y) = \int f d\nu_y^d + \int f d\nu_y^a, \quad \nu_y^d \text{ diffuse, } \nu_y^a \text{ atomic.}$$

In the lattice of all order continuous operators these decompositions correspond to band decompositions: the integral part of T of course belongs to the band L_i of all integral operators (1.1) and the singular part belongs to its orthogonal band L_s ; the atomic part belongs to the band L_a generated by all Riesz homomorphisms (1.2) (see Corollary 6.3), and the diffuse part to its orthogonal band L_d . While there are good characterizations of integral operators and the bands generated by a single Riesz homomorphism [31], the bands L_s , L_a , L_d seem to be less well understood. In this paper we give characterizations of operators in L_s , L_a or L_d , which are similar in spirit to the known results on integral operators and Riesz homomorphisms.

For an operator T with a diffuse representation (i.e. $T \in L_d$), it is clear that (let $d(A)$ be the diameter of a set $A \subset X$ for a metric d on X)

$$(1.6) \quad d(A_n) \rightarrow 0 \text{ implies } T\chi_{A_n} \rightarrow 0 \text{ } \nu\text{-a.e.}$$

We show that (1.6) already characterizes such operators—a result analogous to Buhvalov's characterization of integral operators.

It is well known that an operator T in $L_1(X, \mathcal{A}, \mu)$ is an integral operator if and only if

$$(1.7) \quad \begin{array}{l} \text{every band } L_1(A) \text{ contains a band } L_1(B), A, B \in \mathcal{A}, \text{ such} \\ \text{that } T|_{L_1(B)} \text{ is compact.} \end{array}$$

If we replace the band $L_1(B)$ in (1.7) by a nonatomic sublattice $L_1(\Sigma, \mu)$ (Σ a sub- σ -algebra of \mathcal{A}), then we obtain a characterization of operators in L_d . These descriptions of L_d together with another one derived from the Dunford-Pettis characterization of integral operators are contained in §4.

On the other hand, we can characterize L_s and L_a (in §§ 5 and 6) by weakening appropriately the disjointness-preserving property of Riesz homomorphisms. For example, T has an atomic representation (i.e. $T \in L_a$) if for all L_1 -bounded sequences $f_n \in E$ we have

$$(1.8) \quad f_n \xrightarrow{\mu} 0 \text{ implies } Tf_n \xrightarrow{\nu} 0.$$

and T has a singular representation ($T \in L_s$) if for all such (f_n) we have

$$(1.9) \quad d(\{f_n \neq 0\}) \rightarrow 0 \quad \text{implies} \quad Tf_n \xrightarrow{\nu} 0.$$

Further conditions are inspired by the so-called Maharam property of Luxemburg and Schep [31], and we also express $T \in L_s, L_a$ in terms of the vector-valued martingale determined by T' .

In this context we obtain the following generalization of a result of J. Bourgain on Dunford-Pettis operators: If $T: L_1[0, 1] \rightarrow L_1[0, 1]$ then T' has a diffuse representation if and only if there is a function $G: [0, 1] \rightarrow M[0, 1]$ such that for $f \in L_1[0, 1]$

$$(1.10) \quad Tf = \int G(x)f(x) d\mu(x),$$

where the Bochner integral exists with respect to the so-called 'discrepancy'-norm on $M[0, 1]$ (see Definition 3.8), i.e. $T: L_1[0, 1] \rightarrow M[0, 1]$ is representable with respect to this weaker norm on $M[0, 1]$ (see Corollary 4.4).

In §2 we collect some auxiliary results. In §3 we introduce notions which allow us to replace the rather restrictive conditions on diameters in (1.6) and (1.9) by a more general condition.

Some applications of these characterizations to a result of N. Ghoussoub and M. Talagrand on Banach lattices with the Radon-Nikodym property, to convolution operators and to essential spectra of L_1 -operators will appear in [47].

Now we fix some notations and recall some definitions. Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) always be *measure spaces* where X and Y are separable metric spaces and \mathcal{A}, \mathcal{B} are their σ -algebras of Borel sets, such that (X, \mathcal{A}) and (Y, \mathcal{B}) are standard measure spaces [37, Chapter V, §2]. μ and ν always stand for strictly positive and finite measures. d denotes the metric on X or Y , and $d(A)$ is the diameter of a subset of X or Y . Recall that a *random measure* from (Y, \mathcal{B}) to (X, \mathcal{A}) is a family $(\mu_y)_{y \in Y}$ of finite (not necessarily probability-) measures such that for all $A \in \mathcal{A}$ the function $y \in Y \rightarrow \mu_y(A)$ is \mathcal{B} -measurable. Such a random measure is called *ν -orthogonal* if all μ_y are probability measures and there is a measurable map $\sigma: X \rightarrow Y$ such that $\mu_y(\sigma^{-1}(y)) = 1$ for ν -almost all $y \in Y$. In probability theory these ν -orthogonal random measures appear as regular conditional probability distributions.

$L_0(X, \mathcal{A}, \mu)$ is the space of equivalence classes of μ -measurable functions on X and for $A \in \mathcal{A}$ and a σ -subalgebra Σ of \mathcal{A} we use the abbreviations

$$\mathcal{A}_A = \{A \cap A': A' \in \mathcal{A}\}, \quad L_0(A, \mu) = L_0(A, \mathcal{A}_A, \mu|_{\mathcal{A}_A}),$$

$$E_A = \{f|_A: f \in E\}, \quad \Sigma_A = \{A \cap A': A' \in \Sigma\},$$

$$L_0(\Sigma_A, \mu) = L_0(A, \Sigma_A, \sigma_A, \mu|_{\Sigma_A}), \quad E_\Sigma = \{f \in E: f|_{\Sigma}\text{-measurable}\},$$

\mathcal{E}_{Σ_A} denotes the conditional expectation operator from $L_p(X, \mathcal{A}, \mu)$ onto $L_p(\Sigma_A, \mu)$, $1 \leq p \leq \infty$.

Let $C(X)$ denote the space of all bounded, continuous functions on X and $M(X)$ the space of all signed, finite measures on (X, \mathcal{A}) . δ_x is the point measure at $x \in X$. By the Radon-Nikodym theorem we can identify $L_1(X, \mu)$ with the subspace

$\{\lambda \in M(X): \lambda \ll \mu\}$ of $M(X)$ by $f \rightarrow f \cdot \mu$, where $(f \cdot \mu)(A) = \int_A f d\mu$. Since μ is strictly positive we can identify $C(X)$ with a subspace $(C(X))^\wedge$ of $L_\infty(X, \mathcal{A}, \mu)$ by $g \in C(X) \rightarrow \hat{g} \in L_\infty(X, \mathcal{A}, \mu)$. Let $T: L_\infty(X, \mathcal{A}, \mu) \rightarrow L_\infty(Y, \mathcal{B}, \nu)$ be a w^* -continuous operator with $T((C(X))^\wedge) \subset (C(Y))^\wedge$. Then T defines an operator $\hat{T}: C(X) \rightarrow C(Y)$ such that \hat{T}' is an 'extension' of T' in the sense that

$$(1.11) \quad \hat{T}'(f \cdot \nu) = (T'f) \cdot \mu \quad \text{for } f \in L_1(Y, \nu).$$

Therefore, in many statements below, we will not distinguish between $f, f \cdot \mu; C(X), (C(X))^\wedge$ and T, \hat{T} .

" $\mu_n \xrightarrow{w^*} \lambda$ " denotes the weak convergence of measures, i.e. the $\sigma(M(X), C(X))$ -convergence.

" $f_n \xrightarrow{\mu} 0$ " means that f_n converges in μ -measure to zero.

By a *Riesz function space* E over (X, \mathcal{A}, μ) we mean a sublattice of $L_0(X, \mathcal{A}, \mu)$ such that:

- (i) E is an ideal in $L_0(X, \mathcal{A}, \mu)$,
- (ii) $\chi_A \in E$ for all $A \in \mathcal{A}$ with $\mu(A) < \infty$.

The Köthe dual of E is given by

$$E^\times = \left\{ f \in L_0(X, \mathcal{A}, \mu): \int |f \cdot g| d\mu < \infty \text{ for all } g \in E \right\}.$$

If $1 \in E^\times$, then E^\times is a Riesz function space too.

Let E and F be Riesz function spaces and $T: E \rightarrow F$ be a linear operator. We call T *regular* if for all $f \in E$ there is a $g \in F$ such that $|h| \leq f$ implies $|Th| \leq g$. The class $L_r(E, F)$ of all regular operators is a lattice in the natural ordering [39, IV, §1; 29; 44, VIII, §2] with the modulus of T given by

$$|T|f = \sup\{|Tg|: |g| \leq f\} \quad \text{for } f \in E_+.$$

A regular operator is *order continuous* if $|h_n| \leq f \in E, h_n \rightarrow 0$ a.e. always implies that $Th_n \rightarrow 0$ a.e. The class $L_0(E, F)$ of all order-continuous operators is a band in $L_r(E, F)$ [44, VIII, §3]. If E and F are Köthe function spaces with order-continuous norms then every regular operator is order continuous [44, VIII, §3]; if E and F are both abstract L -space or order complete abstract M -spaces then every bounded operator is regular [39, IV, §1].

Let E and F be Riesz function spaces with $1 \in E^\times$ and $1 \in F^\times$ and let $T: E \rightarrow F$ be order continuous. Then the adjoint $T': F^\times \rightarrow E^\times$ is order continuous again.

For unexplained terminology and relevant information on lattices and regular operators see [30, 32, 39], on probability theory see [36, 37], and on Banach spaces see [25–27].

2. The representation theorem. Important special cases of the following result were shown in [36, §5.4; 1; 19; 20; 41 and 12].

2.1 THEOREM [42]. *Let E and F be Riesz function spaces over (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) respectively. For a regular operator $T: E \rightarrow F$ the following are equivalent:*

- (a) T is order continuous.

(b) *There is a random measure $(\mu_y)_{y \in Y}$ from (Y, \mathcal{B}) to (X, \mathcal{A}) such that for all $f \in E$*

$$(+) \quad Tf(y) = \int f d\mu_y \quad \nu\text{-a.e.}$$

This representation is unique in the sense that if $(\mu'_y)_{y \in Y}$ is a second random measure with $(+)$, then $\mu_y = \mu'_y$ for ν -almost all $y \in Y$.

2.2 REMARK. In $(+)$ it is assumed implicitly that

(i) $\mu(A) = 0$ implies $|\mu_y|(A) = 0$ ν -a.e.,

(ii) $\int |f| d|\mu_y| < \infty$ ν -a.e.

Because of (i) we may use the same symbol for a function and its equivalence class.

2.3 REMARK [42, THEOREM 4.1]. If $S, T: E \rightarrow F$ are order continuous operators and $(\mu_y[T])$ denotes a representation of T then

(a) $\mu_y[T + S] = \mu_y[T] + \mu_y[S]$ ν -a.e.,

(b) $0 \leq T \leq S \Leftrightarrow 0 \leq \mu_y[T] \leq \mu_y[S]$ ν -a.e.,

(c) $\mu_y[T^+] = \mu_y[T]^+, \mu_y[|T|] = |\mu_y[T]|$ ν -a.e.,

(d) $\mu_y[T \wedge S] = \mu_y[T] \wedge \mu_y[S]$ ν -a.e.

The next lemma will be used many times in the following sections. It also provides a proof for Theorem 2.1. (In the following we use identifications of spaces as discussed in (1.11)).

2.4 LEMMA. Let $T: E \rightarrow L_0(Y, \mathcal{B}, \nu)$ be an order-continuous operator. For every $\varepsilon > 0$ there is a compact $K \subset \text{supp } \nu$ with $\nu(Y - K) \leq \varepsilon$ such that:

(a) $\chi_K T: L_\infty(X, \mathcal{A}, \mu) \rightarrow L_\infty(K, \mathcal{B}_K, \nu)$ and $\chi_K T: C(X) \rightarrow C(K)$.

(b) The operator $(\chi_K T)': L_1(K, \nu) \rightarrow L_1(X, \mu)$ "extends" to a w^* -continuous operator $(\chi_K T)': M(K) \rightarrow M(X)$ such that $(\chi_K T')(U_{M(K)})$ is uniformly tight.

(c) $\mu_y := (\chi_K T)' \delta_y, y \in K$, defines a random measure from (K, \mathcal{B}_K) to (X, \mathcal{A}) such that

$$Tf(y) = \int f d\mu_y \quad \nu\text{-a.e. for all } f \in E,$$

$$((\chi_K T)'\lambda)(A) = \int_K \mu_y(A) d\lambda(y) \quad \text{for } \lambda \in M(K), A \in \mathcal{A}.$$

PROOF. (a) $|T|: E \rightarrow L_0(Y, \mathcal{B}, \nu)$ exists and is also order continuous. Choose a sequence of compact sets $X_m \subset X$ with

$$(2.1) \quad X_m \subset X_{m+1} \subset \cdots, \quad \mu\left(X - \bigcup_m X_m\right) = 0,$$

$|\mu|_{X_m}$ is strictly positive

and a sequence (f_n) dense in $\bigcup_m C(X_m)$ with respect to the sup-norm. Since $\chi_{X_m^c} \rightarrow 0$ a.e. it follows that $|T|\chi_{X_m^c} \rightarrow 0$ a.e.

By Egoroff's theorem and [37, II, Lemma 4.2], there is a compact $K \subset Y$ such that:

$$(2.2) \quad \nu(Y - K) \leq \varepsilon, \quad K \subset \text{supp } \nu,$$

$$(2.3) \quad |T|1 \in L_\infty(K, \nu),$$

$$(2.4) \quad \|\chi_K |T| \chi_{X_m^c}\|_\infty \rightarrow 0 \quad \text{for } m \rightarrow \infty,$$

$$(2.5) \quad \text{all } Tf_n|_K \text{ and } |T|f_n|_K \text{ are continuous on } K.$$

By (2.3) the regular operators $T, |T|: L_\infty(X, \mu) \rightarrow L_\infty(K, \nu)$ are continuous with respect to $\|\cdot\|_\infty$ -norm. For every $f \in U_{C(X)}$ there is a sequence (f_{n_m}) such that

$$\|Tf - Tf_{n_m}\|_\infty \leq \| |T| \chi_{X_m^c} \|_\infty + \frac{1}{m} \rightarrow 0 \quad \text{for } m \rightarrow \infty.$$

Therefore, $T: C(X) \rightarrow C(K)$ by (2.5).

(b) The operator $(\chi_K T)': C(K)' = M(K) \rightarrow C(X)'$ actually takes its values in $M(X)$. Indeed, the finitely additive measure $(\lambda_K T)' \lambda, \lambda \in M(K)$, is σ -additive on X_m (cf. [37, II, Theorem 5.7]) and

$$(2.6) \quad |(\chi_K T)' \lambda|(X_0 - X_m) \leq |\lambda|(|T| \chi_{X_m^c}) \leq \|\lambda\| \cdot \|\chi_K |T| \chi_{X_m^c}\|_\infty \xrightarrow{m \rightarrow \infty} 0.$$

$(\chi_K T)'$ is obviously w^* -continuous and (2.6) shows that $(\chi_K T)'(U_{M(K)})$ is uniformly tight. It follows from $\chi_K T: L_\infty(X, \mu) \rightarrow L_\infty(K, \nu)$ and (1.11) that

$$(2.7) \quad \lambda \in M(K), \quad \lambda \ll \nu \Rightarrow (\chi_K T)' \lambda \ll \mu.$$

(c) For all $f \in C(X), y \in K$ and $\lambda \in M(X)$ we have

$$(2.8) \quad Tf(y) = \delta_y(Tf) = T'\delta_y(f) = \int f d\mu_y,$$

$$(\chi_K T)' \lambda(f) = \lambda(\chi_K Tf) = \int_K Tf(y) d\lambda(y) = \int \mu_y(f) d\lambda(y).$$

Hence

$$(2.9) \quad (\chi_K T)' \lambda(A) = \int \mu_y(A) d\lambda(y) \quad \text{for all } A \in \mathcal{A}.$$

By (2.8) the map $y \in K \rightarrow \mu_y \in M(X)$ is continuous from K to $M(X)$ with the w^* -topology. Therefore $(\mu_y)_{y \in K}$ is a random measure such that (cf. (2.7) and (2.9))

$$(2.10) \quad \mu(A) = 0 \quad \text{implies} \quad \mu_y(A) = 0 \quad \nu\text{-a.e.}$$

In order to see that (2.8) holds for $f \in L_\infty(X, \mu)$ we choose a sequence $f_n \in C(X)$ with $f_n \rightarrow f$ μ -a.e. By the order of continuity of T and (2.10) there is a ν -nullset B such that for $y \notin B$

$$Tf_n(y) \rightarrow Tf(y), \quad f_n \rightarrow f \quad \mu_y\text{-a.e.}$$

From (2.8) and the dominated convergence theorem we obtain, for $y \notin B$,

$$Tf(y) = \lim_n Tf_n(y) = \lim \int f_n d\mu_y = \int f d\mu_y.$$

If $f \in E$ then $f_n := f \wedge n1 \nearrow f, f_n \in L_\infty$, and (2.8) follows from a similar argument using Fatou's lemma. \square

2.5 DEFINITION. Let E and F be Riesz function spaces over (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) respectively. Let $L_i(E, F)$ be the class of integral operators from E to F .

By $L_s(E, F)$ ($L_d(E, F)$, $L_a(E, F)$) we denote the class of all operators with a singular (diffuse, atomic) representation, i.e. the class of all operators $T: E \rightarrow F$ such that $\mu_y[T]$ is μ -singular (diffuse, atomic) for ν -almost all $y \in Y$.

2.6 PROPOSITION. $L_i(E, F)$, $L_s(E, F)$, $L_d(E, F)$ and $L_a(E, F)$ are bands in $L_0(E, F)$ and we have the (lattice) orthogonal decompositions

$$L_0(E, F) = L_i(E, F) \oplus L_s(E, F), \quad L_0(E, F) = L_d(E, F) \oplus L_a(E, F).$$

PROOF. The decomposition of a measure λ into its diffuse part λ^d and its atomic part λ^a as well as the Lebesgue decomposition $\lambda = \lambda^i + \lambda^s$ into its μ -absolutely continuous part λ^i and its μ -singular part λ^s define maps $\lambda \rightarrow \lambda^d$, $\lambda \rightarrow \lambda^a$, $\lambda \rightarrow \lambda^i$, $\lambda \rightarrow \lambda^s$ in $M(X)$, which are measurable with respect to the Borel structure generated by the $\sigma(M(X), C(X))$ -topology on $M(X)$ [9]. Therefore, for a random measure $(\mu_y)_{y \in Y}$, $(\mu_y^b)_{y \in Y}$ is again a random measure with $|\mu_y^b| \leq |\mu_y|$, $b \in \{d, a, i, s\}$. By Remark 2.3 the required order properties can be reduced to the corresponding order properties of the classes of μ -singular, diffuse or atomic measures in $M(X)$. \square

Finally we point out a connection between the representation (+) and martingale representations of an operator (cf. [7, Chapter V]).

2.7 DEFINITION. Let E and F be Riesz function spaces over (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) respectively with $1 \in E^\times$, $1 \in F^\times$. Let Σ_n be an increasing sequence of algebras, each generated by a finite member of atoms A_i^n , $i = 1, \dots, s_n$, such that $\bigcup_n \Sigma_n$ generates \mathcal{A} . If $T: E \rightarrow F$ is a linear operator, put

$$\mathcal{F}_n(x) = \sum_{i=1}^{s_n} T(\chi_{A_i^n}) \cdot \mu(A_i^n)^{-1} \cdot \chi_{A_i^n}(x).$$

Then $(\mathcal{F}_n, \Sigma_n)$ is a “martingale determined by T ”.

2.8 PROPOSITION. Let $T: L_1(X, \mathcal{A}, \mu) \rightarrow L_1(Y, \mathcal{B}, \nu)$ be bounded and $(\mathcal{F}_n, \Sigma_n)$ be a martingale determined by T .

- (a) For all $g \in L_\infty(Y, \nu)$ we have $T'g = L_1 - \lim g(\mathcal{F}_n)$.
- (b) If $(\nu_n)_{n \in \mathbb{N}}$ is the representation of T' , then $w^* - \lim \mathcal{F}_n(x) \cdot \nu = \nu_x \mu - a.e.$

PROOF. Since T is L_1 -bounded we have for all n

$$\sup_{x \in X} \|\mathcal{F}_n(x)\|_{L_1(Y, \nu)} \leq \|T\|_{L_1 \rightarrow L_1} < \infty$$

and for all $f \in L_1(X, \mu)$ the Bochner integrals $\int_x \mathcal{F}_n(x) f(x) d\mu(x)$ converge to Tf in the $L_1(Y, \nu)$ -norm.

- (a) For all $g \in L_\infty(Y, \nu)$ and $f \in L_1(X, \mu)$ we have

$$\begin{aligned} T'g(f) &= g(Tf) = g\left(\lim_n \int \mathcal{F}_n(x) \cdot f(x) d\mu(x)\right) \\ &= \lim_n \int g(\mathcal{F}_n(x) \cdot f(x)) d\mu(x) \\ &= \int \left[\lim_n g(\mathcal{F}_n(x))\right] f(x) d\mu(x) \end{aligned}$$

since the scalar-valued martingale $(g \circ \mathcal{F}_n, \Sigma_n)$ converges. Hence $T'g = \lim_n g(\mathcal{F}_n)$.

(b) Choose a sequence $(g_k) \subset C(Y)$ that generates the w^* -convergence in $M(Y)$ (cf. [37, II, Theorem 6.6]). Since the martingale $(g_k \circ \mathcal{F}_n, \Sigma_n)$ converges μ -a.e. to $T'g_k$ and

$$(T'g_k)(x) = \nu_x(g_k) \quad \mu\text{-a.e. for all } k,$$

there is a μ -nullset $N \in \mathcal{A}$ such that for all k and $x \notin N$

$$g_k(\mathcal{F}_n(x)) \rightarrow \nu_x(g_k) \quad \text{for } n \rightarrow \infty. \quad \square$$

3. Universal convergence and U -equi-integrability. The notions to be introduced in this section are analogous to convergence in measure and equi-integrability respectively, in the sense that “smallness in measure” is replaced by a topological condition.

3.1 DEFINITION. A sequence $A_n \in \mathcal{A}$ of Borel sets in X converges to zero universally if every subsequence of A_n contains a subsequence (A_{n_k}) such that $\bigcap_m \overline{\bigcup_{k=m}^\infty A_{n_k}}$ is countable (write $A_n \xrightarrow{u} 0$).

In particular, if $A_n \xrightarrow{u} 0$ then $\mu(A_n) \xrightarrow{u} 0$ for every diffuse measure μ . By [37, Theorem I, 2.8 and II, 8.1], one could also replace the word “countable” in this definition by “universal null set”. A “typical” example of a universal nullsequence is the sequence of dyadic intervals $[(i-1)/2^n, i/2^n]$, $i = 1, \dots, 2^n$, $n \in \mathbb{N}$, or, more generally, every sequence $A_n \in \mathcal{A}$ with $d(A_n) \rightarrow 0$. On the other hand, if A_n is the set that is taken out of $[0, 1]$ at the n th step of the construction of the Cantor set, then A_n is a pairwise disjoint sequence which does not converge to zero universally.

3.2 DEFINITION. (a) A sequence $\mu_n \in M(X)$ goes to zero universally ($\mu_n \xrightarrow{u} 0$), if for every $\varepsilon > 0$ there is a sequence A_n such that $A_n \xrightarrow{u} 0$ and $\overline{\lim}_n |\mu_n|(A_n^c) \leq \varepsilon$.

(b) A subset $U \subset M(X)$ is called u -equi-integrable if for all $A_n \in \mathcal{A}$ with $A_n \xrightarrow{u} 0$ we have $\sup\{|\mu|(A_n) : \mu \in U\} \rightarrow 0$ for $n \rightarrow \infty$.

If all μ_n and $\mu \in U$ are absolutely continuous with respect to a given measure ν and if one replaces “ $A_n \xrightarrow{u} 0$ ” by “ $\nu(A_n) \rightarrow 0$ ” in (a) and (b), then the above conditions become equivalent to convergence in measure and to equi-integrability, respectively. Observe that these definitions depend only on the topology of X , but they can also be expressed in terms of the metric given on X :

3.3 PROPOSITION. Let U be a uniformly tight subset of $M_+(X)$. Then the following statements are equivalent.

(a) U is u -equi-integrable.

(b) $\sup_{\mu \in U} |\mu|(A) \rightarrow 0$ for $d(A) \rightarrow 0$.

(c) Every limit point of U in the w^* -topology of $M(X)$ is a diffuse measure.

PROOF. (a) \Rightarrow (b) is clear, since $d(A_n) \rightarrow 0$ implies $A_n \xrightarrow{u} 0$.

(b) \Rightarrow (c). If $\mu = w^*\text{-lim } \mu_n$ with $\mu_n \in U$ and if A_n is a decreasing sequence of open balls in X with $\bigcap A_n = \{x\}$, then by [37, Theorem II, 6.1]

$$0 \leq \mu(x) \leq \mu(A_k) \leq \liminf_n \mu_n(A_k) \rightarrow 0 \quad \text{for } k \rightarrow \infty.$$

(c) \Rightarrow (a). Otherwise there are sequences $\mu_n \in U$, $A_n \in \mathcal{A}$ and a constant $C > 0$, such that

$$\mu_n(A_n) \geq C, \quad B = \bigcap_n \overline{\bigcup_{m=n} A_m} \text{ is countable.}$$

Since a uniformly tight subset is w^* -compact in $M(X)$ [37, Theorem II, 6.7] we may assume $\mu_n \xrightarrow{w^*} \mu$. For the closed sets $B_n = \overline{\bigcup_{m=n}^\infty A_m}$ we have

$$\mu(B_n) \geq \lim_n \mu_n(B_n) \geq \lim_n \mu_n(A_n) \geq C.$$

But $\mu(B) \geq C$ contradicts the assumption that μ is a diffuse measure. \square

3.4 PROPOSITION. *Let $(\mu_n) \subset M_+(X)$ be a sequence with $\mu_n \xrightarrow{w^*} \mu \in M(X)$. The following statements are equivalent.*

- (a) $\mu_n \xrightarrow{u} 0$.
- (b) For all $\varepsilon > 0$ and $\delta_i > 0$, $i \in \mathbb{N}$, there are finitely many $K_i \in \mathcal{A}$ with $d(K_i) < \delta_i$ and $\lim_n \mu_n(X - \bigcup K_i) \leq \varepsilon$.
- (c) μ is an atomic measure.

PROOF. (a) \Rightarrow (c) Choose a subsequence (μ_{n_k}) and a sequence $A_k \in \mathcal{A}$ such that $\lim_k \mu_{n_k}(A_k^c) \leq 1/i$ and $\bigcap_m \overline{\bigcup_{k \geq m} A_k}$ is countable. Since $C_m = \overline{\bigcup_{k \geq m} A_k^c}$ is open it follows from the w^* -convergence of μ_{n_k} [37, Theorem II, 6.1] that

$$\mu(C_m) \leq \lim_k \mu_{n_k}(C_m) \leq \lim_k \mu_{n_k}(A_k^c) \leq \frac{1}{i}.$$

For $B_i = \bigcup_m C_m$ we have $\mu(B_i) \leq 1/i$ and B_i^c is countable. Repeating this argument for all $i \in \mathbb{N}$ we obtain that μ is zero on $B = \bigcap_i B_i$ and B^c is countable.

(b) \Rightarrow (c). Let μ^d be the diffuse component of μ and $\varepsilon > 0$. Since $\mu^d(A) \rightarrow 0$ for $d(A) \rightarrow 0$ there is a sequence $\delta_i > 0$ such that for all sequences K_i with $d(K_i) < \delta_i$, $i \in \mathbb{N}$, we have $\mu^d(\bigcup_{i \in \mathbb{N}} K_i) \leq \varepsilon$. By assumption, there are finitely many closed K_i with $d(K_i) < \delta_i$ and $\lim_n \mu_n(X - \bigcup_i K_i) \leq \varepsilon$. Using the w^* -convergence of μ_n we find

$$\mu^d(X) \leq \mu(X - \bigcup K_i) + \mu^d(\bigcup K_i) \leq \lim_n \mu_n(X - \bigcup K_i) + \varepsilon \leq 2\varepsilon$$

for all $\varepsilon > 0$. Hence μ is atomic.

(c) \Rightarrow (a). Let $\mu = \sum_{i=1}^\infty \alpha_i \delta_{x_i}$. Given $\varepsilon > 0$ we choose an i_0 such that $\sum_{i=i_0}^\infty \alpha_i \leq \varepsilon$. For $A_m = \bigcup_{i=1}^{i_0} A(x_i, 1/m)$, where $A(x_i, 1/m)$ is the open ball with center x_i and radius $1/m$, we obtain

$$\lim_n \mu_n(A_m) \geq \mu(A_m) > \sum_{i=1}^{i_0-1} \alpha_i \geq \mu(X) - \varepsilon.$$

Choose a sequence (n_m) with $n_m < n_{m+1}$ so that $\mu_n(A_m) \geq \mu(X) - \varepsilon$ for $n \geq n_m$, and put $B_n = A_m$ if $n_m \leq n < n_{m+1}$. Then $B_n \xrightarrow{u} 0$ and furthermore, we have for $n_m \leq n < n_{m+1}$

$$\mu_n(B_n^c) = \mu_n(A_m^c) \leq \mu_n(X) - \mu(X) + \varepsilon,$$

where $\mu_n(X) - \mu(X) \rightarrow 0$ for $n \rightarrow \infty$.

(c) \Rightarrow (b) is shown similarly. \square

3.5 PROPOSITION. *If $\mu_n \in M(X)$ is u -equi-integrable and a universal null sequence then $\|\mu_n\|_{M(X)} \rightarrow 0$.*

This follows immediately from the definitions. On the other hand we have the following decomposition result whose analogue for convergence in measure and equi-integrability goes back to Kadec and Pełczyński [18, Proof of Theorem 5].

3.6 PROPOSITION. *Let (μ_n) be a bounded, uniformly tight subsequence of $M_+(X)$. Then there is a subsequence (n_k) and sequences $(\nu_k), (\lambda_k) \subset M_+(X)$ such that:*

- (i) $\mu_{n_k} = \nu_k + \lambda_k, \nu_k \wedge \lambda_k = 0$.
- (ii) $\nu_k \xrightarrow{u} 0$ and (λ_k) is u -equi-integrable.

PROOF. Let $D(\cdot, \cdot)$ be a metric on $U_{M_+(X)}$ generating the $\sigma(M(X), C(X))$ -topology (cf. [37, Theorem II, 6.2]). Since μ_n is uniformly tight, the sequence is relatively D -compact [37, Theorem II, 6.7] and we may assume that μ_n D -converges to some $\mu \in M_+(X)$. If μ^a and μ^d denotes the atomic part and the diffuse part of μ respectively, then it is enough to find Borel sets A_k and a subsequence (n_k) such that $\mu_{n_k}|_{A_k} \xrightarrow{D} \mu^a$. Indeed, according to 3.3 and 3.4, $\nu_k = \mu_{n_k}|_{A_k}$ and $\lambda_k = \mu_{n_k}|_{A_k^c}$ have all the required properties.

Let $\mu^a = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$. Since every ball K in X contains a ball K_1 with $\mu(\bar{K}_1 - \dot{K}_1) = 0$ (cf. [37, Proof of Lemma II, 6.5]), there is for every x_i a decreasing sequence of balls $(K_i^m)_{m \in \mathbb{N}}$ with

$$\{x_i\} = \bigcap_m K_i^m, \quad \mu_n(K_i^m) \rightarrow \mu(K_i^m) \quad \text{for all } m, i.$$

Given some $\varepsilon > 0$ we choose i_0 such that $D(\mu^a, \sum_{i=1}^{i_0} \alpha_i \delta_{x_i}) \leq \varepsilon/2$. For $B^m = \bigcup_{i=1}^{i_0} K_i^m$ we have

$$\mu_n|_{B^m} \xrightarrow{D} \mu|_{B^m} \quad \text{for } n \rightarrow \infty$$

and (by the dominated convergence theorem)

$$\mu|_{B^m} \xrightarrow{D} \mu|_{\bigcap_m B^m} = \sum_{i=1}^{i_0} \alpha_i \delta_{x_i}.$$

Therefore, there are m_0 and n_0 such that $D(\mu_{n_0}|_{B_0^m}, \sum_{i=1}^{i_0} \alpha_i \delta_{x_i}) \leq \varepsilon/2$. Hence for all $\varepsilon > 0$ we can find a set A_0 (put $A_0 = B^{m_0}$) and n_0 such that $D(f_{n_0}|_{A_0}, \mu^a) \leq \varepsilon$. This completes the proof of 3.6. \square

3.7 REMARK. We found it convenient to define u -convergence in topological terms. From a more ‘abstract’ point of view the concept can be described as follows: Let \mathcal{A} be a countably generated σ -algebra of subsets of some set Ω and let $\mathcal{C} \subset \mathcal{A}$ be a countable generating system, which is closed under finite intersections and unions. Then for $A_n \in \mathcal{A}$, $A_n \xrightarrow{u} 0$ ‘means’ that every subsequence of (A_n) has a subsequence (A_{n_m}) such that

$$(3.1) \quad \bigcap \left\{ C \in \mathcal{C} \mid \exists k \text{ such that } \bigcup_{m=k}^{\infty} A_{n_m} \subset C \right\}$$

is countable. Indeed, by replacing (Ω, \mathcal{A}) by its canonical separable Borel space (cf. [37, V.2]) we may assume $\{x\} \in \mathcal{A}$ for $x \in \Omega$ and then \mathcal{C} is a base of the closed subsets of a metrizable topology on Ω such that (3.1) corresponds to $\bigcap_k \overline{\bigcup_{m=k}^{\infty} A_{n_k}}$ (by Urysohn's metrization theorem [2, p. 125]). \square

Next we recall a norm on the vector space $M[0, 1]$ which is called "discrepancy"-norm in mathematical physics (cf. [35]). In the context of representation theory of linear operators it was first used by J. Bourgain [3].

3.8 DEFINITION. M_0 denotes the vector space $M[0, 1]$ normed by $\|\mu\|_0 = \sup_{I \in J} |\mu(I)|$, where J is the set of all intervals in $[0, 1]$.

3.9 REMARK. M_0 is a noncomplete normed space. If (μ_n) is bounded in $M[0, 1]$ and $\mu_n \xrightarrow{\|\cdot\|_0} \mu \in M[0, 1]$ then $\mu_n \xrightarrow{w^*} \mu$. If, in addition, all μ_n are diffuse, then so is μ .

3.10 PROPOSITION. Let U be a $M[0, 1]$ -bounded set of diffuse measures. If U is u -equi-integrable, then U is relatively compact in M_0 . If all the measures in U are positive then the converse is true too.

PROOF. " \Rightarrow " Since U is w^* -compact it is enough to show that $\mu_n \in U$, $\mu_n \xrightarrow{w^*} \mu$ implies $\|\mu_n - \mu\|_0 \rightarrow 0$. Since (μ_n) is u -equi-integrable and μ is diffuse by Proposition 3.3 there is for every $\varepsilon > 0$ some $\delta > 0$ such that

$$(3.2) \quad d(I) < \delta \Rightarrow |\mu|(I) \leq \varepsilon, \quad |\mu_n|(I) \leq \varepsilon \quad \text{for all } n.$$

Furthermore, we can choose finitely many intervals I_1, \dots, I_r such that for every interval with $d(I) \geq \delta$ there is some $s \in (1, \dots, r)$ and intervals J_1, J_2 with

$$(3.3) \quad I_s \Delta I \subset J_1 \cup J_2, \quad d(J_1) \leq \delta, \quad d(J_2) \leq \delta.$$

Let I be an arbitrary interval in $[0, 1]$. Depending on whether $d(I) < \delta$ or $d(I) \geq \delta$ we have by (3.2)

$$|\mu_n(I) - \mu(I)| \leq |\mu_n|(I) + |\mu|(I) \leq 2\varepsilon$$

or by (3.2) and (3.3)

$$|\mu_n(I) - \mu(I)| \leq 4\varepsilon + \max_{i=1}^r |\mu_n(I_i) - \mu(I_i)|.$$

Since every interval is a convergence set for the diffuse measure μ , it follows that $\|\mu_n - \mu\|_0 \leq 5\varepsilon$ for n large enough.

" \Leftarrow " Let $\varepsilon > 0$. By assumption there are finitely many μ_1, \dots, μ_r with

$$\min_{i=1}^r \|\mu - \mu_i\|_0 \leq \varepsilon \quad \text{for all } \mu \in U.$$

Therefore there is a $\delta > 0$, such that for all $\mu \in U$

$$\mu(I) \leq \varepsilon + \max_{i=1}^r |\mu_i(I)| \leq 2\varepsilon \quad \text{for } d(I) < \delta.$$

Since $\mu \geq 0$ the proof is complete. \square

4. Operators with a diffuse representation. Recall the following characterizations of an integral operator $T: E \rightarrow F$:

—(Buhvalov [5], [40], [45]) For all $f \in E_+$ and $A_n \in \mathcal{A}$ with $\mu(A_n) \rightarrow 0$ we have $T(\chi_{A_n} f) \rightarrow 0$ ν -a.e.

—(Dunford and Pettis [10]) For all $\varepsilon > 0$ there is an $A \in \mathcal{A}$ with $\mu(A^c) \leq \varepsilon$ such that $T(U_{L_1(A, \mu)} \cap E)$ is equi-integrable.

—(cf. [7, §V.2]) For all $\varepsilon > 0$ there is an $A \in \mathcal{A}$ with $\mu(A^c) \leq \varepsilon$ such that a martingale determined by $T\chi_A$ (see 2.7) converges in $L_\infty(A, \mu, L_1(Y, \nu))$.

By replacing convergence in measure, equi-integrability and the $L_2(L_1)$ -norm by the corresponding ‘universal’ notions from §3, we obtain characterizations of operators with a diffuse representation. However, we have to formulate some of the conditions for the adjoint operator. The reason is that it is a self-dual property for a regular operator to be an integral operator, while to have a diffuse representation it is not.

4.1 EXAMPLE. Let μ be the Lebesgue-measure on $[0, 1]$. The operator

$$T: L_1([0, 1]^2, \mu \otimes \mu) \rightarrow L_1([0, 1], \mu), \quad Tf(y) = \int f(x, y) d\mu(x)$$

has the diffuse representation $\mu_y = \mu \otimes \delta_y$ while T' has the atomic representation $\nu_{(x, y)} = \delta_y$.

As an example for the σ -algebra appearing in Theorem 4.2(b), we could choose here the Borel sets of the form $A \times [0, 1]$, $A \subset [0, 1]$.

4.2 THEOREM. Let E and F be Riesz function spaces on (X, \mathcal{A}, μ) or (Y, \mathcal{B}, ν) respectively with $1 \in E^\times$, $1 \in F^\times$. For an order continuous operator the following are equivalent.

- (a) T has a diffuse representation.
- (b) For all $A \in \mathcal{A}$, $\mu(A) > 0$, there is a σ -subalgebra Σ of \mathcal{A}_A , such that $\mu|_\Sigma$ is diffuse and $T|_{E_\Sigma}$ is an integral operator.
- (c) For all $f \in E_+$ and $A_n \in \mathcal{A}$ with $A_n \xrightarrow{u} 0$ we have $T(\chi_{A_n} \cdot f) \rightarrow 0$ ν -a.e.
- (d) For all $\varepsilon > 0$ there is a $B \in \mathcal{B}$ with $\nu(B^c) \leq \varepsilon$ such that $T'(U_{L_1(B, \nu)} \cap F^\times)$ is u -equi-integrable.
- (e) (Assume in addition that $X = [0, 1]$.) For every $\varepsilon > 0$ there is a $B \in \mathcal{B}$ with $\mu(B^c) \leq \varepsilon$ such that a martingale determined by $T'\chi_B: F^\times \rightarrow E^\times$ (see 2.7) converges in $L_\infty(Y, \nu, M_0)$ (see 3.8).

PROOF. By 2.4 we can choose compact sets $X_m \subset X$ and $Y_m \subset Y$ such that:

$$(4.1) \quad X_m \subset X_{m+1} \subset \dots, \quad \mu(X - \bigcup X_m) = 0, \quad \mu|_{X_m} \text{ is strictly positive,}$$

$$(4.2) \quad Y_m \subset Y_{m+1} \subset \dots, \quad \nu(Y - \bigcup Y_m) = 0, \quad \nu|_{Y_m} \text{ is strictly positive,}$$

$$(4.3) \quad \chi_{Y_n} T: C(X) \rightarrow C(Y_n), \quad \chi_{X_m} T': L_\infty(Y, \nu) \rightarrow L_\infty(X_m, \mu),$$

$$(4.4) \quad T'\chi_{Y_n}: L_1(Y_n, \nu) \rightarrow L_1(X, \mu) \text{ ‘extends’ to a } w^*\text{-continuous map } (\chi_{Y_n} T)': M(Y_n) \rightarrow M(X), \text{ and}$$

$$(4.5) \quad T'(U_{M(Y_n)}) \text{ is uniformly tight in } M(X) \text{ for all } n.$$

By $(\mu_y)_{y \in Y}$ we denote the representation of T .

(a) \Rightarrow (b) is essentially shown in [46]: for every random measure $(\mu_y)_{y \in Y}$ with diffuse measures μ_y on (A, \mathcal{A}_A) , there is a σ -subalgebra Σ of \mathcal{A}_A such that μ_y is μ -absolutely continuous on Σ for ν -almost all $y \in Y$.

(b) \Rightarrow (a). If the atomic part of T were not zero, then by (4.3) and [19, Theorem 5.5] there would be an $A \in \mathcal{A}$ such that $T\chi_A$ is an isomorphism as an operator from $L_1(A)$ to $L_1(Y, \nu)$. This contradicts (b).

(a) \Rightarrow (c). For all $y \in Y$ for which μ_y is diffuse and f is $|\mu_y|$ -integrable we have $f_n = \chi_{A_n} f \rightarrow 0$ in μ_y -measure and by the dominated convergence theorem

$$|Tf_n(y)| = |\mu_y(f_n)| \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

(c) \Rightarrow (d). First we show that T^+ has property (c) too. Given $A_n \in \mathcal{A}$, $A_n \xrightarrow{u} 0$, we have for $h_n = f \cdot \chi_{A_n^c}$

$$(4.6) \quad \limsup T^+ h_n \leq T^+ f.$$

On the other hand, for every $0 \leq h \leq f$ we have $h \wedge h_n = h \cdot \chi_{A_n^c}$ and therefore $T(h \wedge h_n) \rightarrow Th$ ν -a.e. This implies

$$(4.7) \quad T^+ f = \sup \{Th : 0 \leq h \leq f\} \leq \liminf T^+(h_n).$$

From (4.6) and (4.7) we obtain $T^+ f = \lim T^+ h_n$, i.e. $T^+(f \cdot \chi_{A_n}) \rightarrow 0$ ν -a.e. Hence T^+ , T^- and $|T|$ have property (c).

Given $\varepsilon > 0$ we choose as K_0 one of the Y_n such that $\nu(K_0^c) \leq \varepsilon/2$. From (4.4) we get $(\chi_{K_0} T)^\vee : L_1(K_0, \nu) \rightarrow L_1(X, \mu)$ and by (4.5) we may assume that, in addition to (4.1), the X_m satisfy

$$(4.8) \quad \sup \left\{ \int_{X - X_m} |T|' f d\mu : f \in U_{L_1(K_0, \nu)} \right\} \xrightarrow{m \rightarrow \infty} 0.$$

If for every m we find a compact $K_m \subset K_0$ with $\nu(K_0 - K_m) \leq \varepsilon/2^{m+1}$ such that

$$(4.9) \quad \sup \left\{ \int_{A \cap X_m} |T|' f d\mu : f \in U_{L_1(K_m, \nu)} \right\} \rightarrow 0 \quad \text{for } d(A) \rightarrow 0$$

then $B = \bigcap_m K_m$ satisfies property (d) because of (4.8), (4.9) and 3.3. For a fixed X_m we choose open sets A_i and a sequence i_k such that

$$(4.10) \quad X_m \subset \bigcup_{i=i_k+1}^{i_{k+1}} A_i \quad \text{for all } k,$$

$$(4.11) \quad \max \{d(A_i) : i = i_k + 1, \dots, i_{k+1}\} \rightarrow 0 \quad \text{for } k \rightarrow \infty.$$

Since $A_i \xrightarrow{u} 0$, it follows that $|T|\chi_{A_i} \rightarrow 0$ ν -a.e. By Egoroff's theorem we find a $K_m \subset K_0$ with $\mu(K_0 - K_m) \leq \varepsilon/2^{m+1}$ and

$$(4.12) \quad \|\chi_{K_m} |T| \chi_{A_i}\|_\infty \rightarrow 0 \quad \text{for } i \rightarrow \infty.$$

By Lebesgue's covering lemma [22, p. 154] there are $\delta_k > 0$ such that each $A \subset X_m$ with $d(A) < \delta_k$ is already contained in one of the sets A_i , $i = i_k + 1, \dots, i_{k+1}$.

Therefore, for every $A \subset X_m$ with $d(A) \leq \delta_k$ and $f \in U_{L_1(K_m, \nu)}$ we have

$$\begin{aligned} \int_A |T'|(|f|) d\mu &\leq \max_{i=i_k+1}^{i_k-1} \int_{K_m} |f(y)| |T|(\chi_{A_i})(y) d\nu(y) \\ &\leq \max_{i=i_k+1}^{i_k-1} \|\chi_{K_m} |T| \chi_{A_i}\|_\infty. \end{aligned}$$

For $k \rightarrow \infty$, (4.9) follows from (4.11) and 3.3.

(d) \Rightarrow (a). We may assume that the Y_n in addition to (4.2)–(4.4) satisfy

$$(4.13) \quad T'(U_{L_1(Y_n, \nu)}) \text{ is } u\text{-equi-integrable for all } n$$

(compare (1.11)).

For a fixed $y \in Y_n$ we consider the sequence $\nu_m = \mu(A_m)^{-1} \chi_{A_m} \cdot \nu$, where A_m is a neighborhood base of y in Y_n . Then $\mu_m \xrightarrow{w^*} \delta_y$ and $T'\mu_m \xrightarrow{w^*} T'\delta_y = \mu_y$. Since μ_y is a w^* -limit point of $T'(U_{L_1(Y_n, \nu)})$ it has to be diffuse by (4.13) and 3.3.

(d) \Rightarrow (f). By (4.4) and (d) there is, for every $\varepsilon > 0$, a $B \in \mathcal{B}$, $\nu(B^c) \leq \varepsilon$, such that $V = T'(U_{L_1(B, \nu)})$ is u -equi-integrable. By 3.10, V is compact as a subset of M_0 . But a martingale $(\mathcal{F}_n', \Sigma_n)$ determined by $(\chi_B T)'$ takes its values in V and therefore it converges in M_0 .

(f) \Rightarrow (a). If $(\mathcal{F}_n', \Sigma_n)$ is determined by $(\chi_B T)'$ then $\mathcal{F}_n'(y)$ w^* -converges to μ_y for ν -a.e. $y \in B$ according to 2.8(b). Since $(\mathcal{F}_n', \Sigma_n)$ converges in $L_1(B, \nu, M_0)$ these w^* -limits are diffuse ν -a.e. in B by the pointwise convergence theorem for martingales and Remark 3.9. \square

In the characterizations of integral operators quoted above one cannot replace the L_1 -norm by a norm given on E . For example, for $E = L_2$ there are norm-compact, positive convolution operators which are not integral operators (cf. [7, p. 93; 14]). But at least in some cases this can be done for the characterization of diffuse representations.

4.3 COROLLARY. *For a regular operator $T: L_p(X, \mathcal{A}, \mu) \rightarrow L_p(X, \mathcal{A}, \mu)$, $1 < p < \infty$, the following are equivalent.*

- (a) *T has a diffuse representation.*
- (b') *For all $A \in \mathcal{A}$, $\mu(A) > 0$, there is a σ -subalgebra Σ of \mathcal{A}_A such that $\mu|_\Sigma$ is diffuse and $T|_{L_p(\Sigma, \mu)}$ is norm-compact in L_p .*
- (d') *For all $\varepsilon > 0$ there is a $B \in \mathcal{A}$ with $\nu(B^c) \leq \varepsilon$ such that $\|\chi_B T \chi_A\|_{L_p} \rightarrow 0$ for $d(A) \rightarrow 0$.*

PROOF. By [45, Theorem 2.2], there is a density $g \in L_1(X, \mu)$ such that (with $\rho_p(f) = g^{-1/p} \cdot f$)

$$T = \rho_p T \rho_p^{-1}: L_q(X, g \cdot \mu) \rightarrow L_q(X, g \cdot \mu) \quad \text{for all } 1 \leq q \leq \infty.$$

Since T has one of the above properties if and only if \tilde{T} has it we may just as well assume

$$T: L_q(X, \mu) \rightarrow L_q(X, \mu) \quad \text{for all } 1 \leq q \leq \infty.$$

(a) \Rightarrow (b'). By 4.2 there is a σ -subalgebra Σ of \mathcal{A} such that $T|_{L_1(\Sigma, \mu)}$ is an integral operator which we may assume to be compact by [7, §III.2, Proposition 21]. Since $T: L_\infty(\Sigma, \mu) \rightarrow L_\infty(X, \mu)$ it follows by interpolation [23, Chapter 1, Theorem 3.10], that $T|_{L_p(\Sigma, \mu)}$ is compact too.

(a) \Rightarrow (d'). By 4.2 there is for every $\varepsilon > 0$ a $B \in \mathcal{A}$ with $\mu(B^c) \leq \varepsilon$, such that $T'U_{L_1(B, \mu)}$ is u -equi-integrable, i.e. by 3.3

$$\|\chi_B T \chi_A\|_{L_\infty} = \|\chi_A T' \chi_B\|_{L_1} \rightarrow 0 \quad \text{for } d(A) \rightarrow 0.$$

By interpolation we have

$$\begin{aligned} \|\chi_B T \chi_A\|_{L_p} &\leq C \|\chi_B T \chi_A\|_{L_\infty}^{1/p} \cdot \|\chi_B T \chi_A\|_{L_1}^{1/p'} \\ &\leq C_1 \|\chi_B T \chi_A\|_{L_\infty}^{1/p} \rightarrow 0 \quad \text{for } d(A) \rightarrow 0. \end{aligned}$$

(d') \Rightarrow (a). Assume that T has (d') but not a diffuse representation. By (a) \Rightarrow (d') there is a $B \in \mathcal{A}$, $\mu(B) > 0$, such that the atomic part S of $\chi_B T$ is nonzero and satisfies $\|S \chi_A\|_{L_p} \rightarrow 0$ for $d(A) \rightarrow 0$. But since S is atomic this contradicts the fact that for some $A_0 \in \mathcal{A}$, $\mu(A_0) > 0$, $S \chi_{A_0}$ is an isomorphism into $L_p(X, \mu)$ (use the proof of [19, Theorem 5.4] and observe that by interpolation the claim of Lemma 5.2 in [19] is also true for L_p -norm).

(b') \Rightarrow (a) follows similarly to (b) \Rightarrow (a) in 4.2. \square

Denote by I and J the operators

$$I: L_1[0, 1] \rightarrow C[0, 1], \quad If(y) = \int_0^y f(z) d\nu(z),$$

$$J: L_1[0, 1] \rightarrow M_0, \quad Jf = f \cdot \nu$$

(ν is the Lebesgue measure on $[0, 1]$). J. Bourgain has shown in [3, Theorem 7 and Corollary 8] that for a Dunford-Pettis operator $T: L_1[0, 1] \rightarrow L_1[0, 1]$ the operators IT , JT are representable, i.e. there is a function $\mathcal{F} \in L_\infty(X, \mu, C[0, 1])$ (or $L_\infty(X, \mu, M_0)$) such that the Bochner integral

$$(*) \quad Tf = \int_X \mathcal{F}(x) f(x) d\mu(x)$$

exists for all $f \in L_1(X, \mathcal{A}, \mu)$ (cf. [7, §III.1]). This result is contained in

4.4 COROLLARY. *For a bounded linear operator $T: L_1(X, \mathcal{A}, \mu) \rightarrow L_1[0, 1]$ the following are equivalent.*

- (a) IT is representable as in (*).
- (b) JT is representable as in (*).
- (c) T' has a diffuse representation.

A Dunford-Pettis operator has these properties.

PROOF. (b) \Leftrightarrow (c) follows from 4.2. (a) \Leftrightarrow (c) with $E = L_\infty[0, 1]$, $F = L_\infty(X, \mathcal{A}, \mu)$ because a martingale $(\mathcal{F}_n, \Sigma_n)$ determined by T converges in $L_\infty(X, \mu, M_0)$ if and only if T has a representation (*) with $\mathcal{F} = \lim \mathcal{F}_n$ (cf. [7, §V.2]).

(a) \Leftrightarrow (b). For all $f \in L_1[0, 1]$ and each interval $I = [a, b]$ in $[0, 1]$ we have

$$\left| \int_I f(y) d\nu(y) \right| \leq \left| \int_0^a f(y) d\nu(y) \right| + \left| \int_0^b f(y) d\nu(y) \right|.$$

Therefore $\|If\|_{C[0,1]} \leq \|Jf\|_{M_0} \leq 2\|Jf\|_{C[0,1]}$ and (b) is just a reformulation of (a).

Now we assume that T is a Dunford-Pettis operator such that T' does not have a diffuse representation. There is a $B \subset [0, 1]$ such that $(\chi_B T)'$ can be extended to an operator $L_1(B, \nu) \rightarrow L_1(X, \mu)$ (let $\chi_B|T|1 \in L_\infty[0, 1]$) which is even an isomorphism (by [19, Theorem 5.5]). Consequently, if $f_n \in L_\infty(B)$ is a sequence with $C^{-1} \leq \|f_n\|_{L_1} \leq \|f_n\|_{L_\infty} \leq C$ and $f_n \rightarrow 0$ in $\sigma(L_1, L_\infty)$ then $g_n = T'f_n$ has the same properties. We obtain

$$|f_n(Tg_n)| = |T'f_n(g_n)| = \int |g_n|^2 d\mu \geq \|g_n\|_{L_1}^2 \geq C_1^{-2} > 0.$$

On the other hand, since $g_n \rightarrow 0$ in $\sigma(L_1, L_\infty)$, $\|g_n\|_{L_\infty} \leq C$, it follows from the Dunford-Pettis property of T that

$$|f_n(Tg_n)| \leq \|f_n\|_\infty \cdot \|Tg_n\|_1 \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

This contradiction completes the proof. \square

4.2(b) together with a result of Enflo and Starbird [11] gives the following alternative (in the following denote by Σ or Σ' a countably generated σ -subalgebra of algebras of the form \mathcal{A}_A , $A \in \mathcal{A}$, such that $\mu|_\Sigma, \mu|_{\Sigma'}$ are diffuse).

4.5 COROLLARY. *For every bounded linear operator $T: L_1(X, \mathcal{A}, \mu) \rightarrow L_1(Y, \mathcal{B}, \nu)$ and every Σ as above there is either a $\Sigma' \subset \Sigma$ such that $T|_{L_1(\Sigma', \mu)}$ is an isomorphism or there is a $\Sigma' \subset \Sigma$ such that $T|_{L_1(\Sigma', \mu)}$ is compact.*

PROOF. Depending on whether $T: L_1(\Sigma, \mu) \rightarrow L_1(Y, \nu)$ has a diffuse representation or not we may apply 4.2(b) or [19, Theorem 5.5]. To obtain a criterium that distinguishes the alternatives consider a system of sets $A_i^n \in \Sigma$, $i = 1, \dots, 2^n$, $n \in \mathbb{N}$, with

- (i) $A_1^n, \dots, A_{2^n}^n$ is a partition of A (if $\Sigma \subset \mathcal{A}_A$),
- (ii) $A_i^n = A_{2i-1}^{n+1} \cup A_{2i}^{n+1}$, $\mu(A_i^n) = 2^{-n}$ and

$$\alpha_n := \left\| \max_{i=1}^{2^n} |T\chi_{A_i^n}| \right\|_{L_1(Y, \nu)}.$$

If $\alpha_n \not\rightarrow 0$, then it follows from [11] that $T|_{L_1(\Sigma', \mu)}$ is an isomorphism for some $\Sigma' \subset \Sigma$. If $\alpha_n \rightarrow 0$, use [46] to obtain a $\Sigma' \subset \Sigma$ such that $T|_{L_1(\Sigma', \mu)}$ is compact. \square

4.6 REMARK. According to Starbird [43] an operator $T: L_1(X, \mu) \rightarrow L_1(X, \mu)$ is an Enflo operator, if and only if there is a subspace E of $L_1(X, \mu)$ isomorphic to $L_1(X, \mu)$ such that $T|_E$ is an isomorphism. From 4.5, [11] and the known characterization of sublattices of L_1 (cf. [25, §17, Theorem 3]) we obtain

(a) T is not an Enflo operator if and only if every nonatomic sublattice E contains a nonatomic sublattice $F \subset E$ such that $T|_F$ is compact.

This can be compared with the following known results:

(b) T is an integral operator if and only if each band E of $L_1(X, \mu)$ contains a band F such that $T|_F$ is compact [7, §III.2, Proposition 21].

(c) T is weakly compact if and only if every infinite-dimensional subspace E contains an infinite-dimensional subspace F such that $T|_F$ is compact (Pełczyński [38]). \square

A further application of 4.2(b) concerning a result of Doss [8; 15, §7.5] in harmonic analysis is given in [46].

5. Operators with a singular representation. Special examples of operators with a singular representation are Riesz homomorphisms (cf. [39, II.2, III.9; 32, I.7]) and their duals. This is obvious from the following characterizations ((A) \Leftrightarrow (C) \Leftrightarrow (D')) are well known; see [31, 28, 16]).

5.1 PROPOSITION. *Let E and F be Riesz function spaces over (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) respectively with $1 \in E^\times$, $1 \in F^\times$. For an order-continuous operator $T: E \rightarrow F$ the following are equivalent.*

(A) *There are measurable maps $\sigma: Y \rightarrow X$, $a: Y \rightarrow \mathbf{R}$ such that for all $f \in E$*

$$Tf(y) = a(y) \cdot f(\sigma(y)) \quad \nu\text{-a.e.}$$

(C) *T is a Riesz homomorphism, i.e. $|f| \wedge |g| = 0$ implies $|Tf| \wedge |Tg| = 0$ for all $f, g \in E$.*

(D) *For every $\varepsilon > 0$ there is a $B \in \mathcal{B}$, $\nu(B^c) \leq \varepsilon$ such that for every sequence $B_n \subset B$ with $d(B_n) \rightarrow 0$ there is a sequence $A_n \in \mathcal{A}$ with $d(A_n) \rightarrow 0$ and $|T|(\chi_{A_n}) \geq \chi_{B_n} \cdot |T|(1)$.*

(C') *For every $\varepsilon > 0$ there is a $B \in \mathcal{B}$, $\nu(B^c) \leq \varepsilon$ such that for every sequence $f_n \in F_B^\times$ with $d(\text{supp } f_n) \rightarrow 0$ we have $d(\text{supp } T'f_n) \rightarrow 0$.*

(D') *$T': F^\times \rightarrow E^\times$ has the Maharam property [31], i.e. for all $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ such that $|T'|(\chi_B) = \chi_A |T'| (1)$.*

(A') *There is a μ -orthogonal random measure $(\nu_x)_{x \in X}$ on (Y, \mathcal{B}) and a measurable $b: Y \rightarrow \mathbf{R}$ such that for all $f \in F^\times$*

$$T'f(x) = \int_{\sigma^{-1}(x)} f \cdot b \, d\nu_x \quad \mu\text{-a.e.},$$

where $\sigma: Y \rightarrow X$ is the separating map belonging to $(\nu_x)_{x \in X}$.

PROOF. (C) \Leftrightarrow (D'). See [28, 31].

(C) \Rightarrow (A) is shown in [16] by a lifting argument. Alternatively, if $(\mu_y)_{y \in Y}$ is the representation of T and $A_n \in \mathcal{A}$ a sequence that generates \mathcal{A} , then there is a ν -nullset $N \subset \mathcal{B}$ such that

$$(5.1) \quad A_n \cap A_m = \emptyset \Rightarrow \{y: \mu_y(A_n) \neq 0\} \cap \{y: \mu_y(A_m) \neq 0\} \subset N.$$

Therefore, for every $y \in N^c$ the support of μ_y is a single point. Otherwise there were $y_0 \in N^c$, A_{n_0} and A_{m_0} with $A_{n_0} \cap A_{m_0} = \emptyset$ and $\mu_{y_0}(A_{n_0}) > 0$, $\mu_{y_0}(A_{m_0}) > 0$ contradicting (5.1). Now choose $\sigma(y) = \text{supp}(\mu_y)$ and $a(y)$ such that $\mu_y = a(y)\delta_{\sigma(y)}$.

(A) \Rightarrow (A'). Let $(\lambda_x)_{x \in X}$ be the representation of T' and $A_n \in \mathcal{A}$ a sequence that generates \mathcal{A} . For $B_n = \sigma^{-1}(A_n)$ we have

$$\begin{aligned} \int_{A_n} |\lambda_x|(Y - B_n) d\mu(x) &= (|T'| \chi_{Y-B_n}) \chi_{A_n} = \chi_{Y-B_n} (|T| \chi_{A_n}) \\ &= \int \chi_{Y-B_n} |a| \cdot \chi_{\sigma^{-1}(A_n)} d\nu = 0. \end{aligned}$$

Hence there is a μ -null set N such that

$$|\lambda_x|(Y - B_n) = 0 \quad \text{for all } x \in A_n, x \notin N.$$

For a given $x \notin N$ there is a subsequence A_{n_m} with $A_{n_m} \supset A_{n_{m+1}}$, $\{x\} = \bigcap A_{n_m}$, $\sigma^{-1}(x) = \bigcap B_{n_m}$ so that

$$|\lambda_x|(Y - \sigma^{-1}(x)) \leq \lim_m |\lambda_x|(Y - B_{n_m}) = 0.$$

Now put $\nu_x = |\lambda_x|/\|\lambda_x\|$ for $\lambda_x \neq 0$ and

$$b(y) = \|\lambda_{\sigma(y)}\| \cdot \left(\frac{d\lambda_{\sigma(y)}}{d|\lambda_{\sigma(y)}|} \right)(y).$$

(A') \Rightarrow (D'). For a given $A \in \mathcal{A}$ choose $B = \sigma^{-1}(A)$ if $\sigma: Y \rightarrow X$ is the measurable map corresponding to $(\nu_x)_{x \in X}$.

(A) \Rightarrow (D). Choose a compact $B \in \mathcal{B}$, $\nu(B^c) \leq \varepsilon$, such that $\sigma|_B$ is continuous. If $d(B_n) \rightarrow 0$, $B_n \subset B$, then $d(\sigma(B_n)) \rightarrow 0$. Put $A_n = \sigma(B_n)$.

(D) \Rightarrow (C'). Choose a B as in (D). If $B_n = \text{supp}(f_n)$ and $d(B_n) \rightarrow 0$ then there are A_n with $d(A_n) \rightarrow 0$ and $|T|(\chi_{A_n}) \geq \chi_{B_n} \cdot |T|(1)$. Therefore

$$\begin{aligned} \int |T'|(|f_n|) d\mu &= \int |f_n| \chi_{B_n} \cdot |T|(1) d\nu \leq \int |f_n| \cdot |T|(\chi_{A_n}) d\nu \\ &= \int_{A_n} |T'| |f_n| d\mu. \end{aligned}$$

Hence $\text{supp}(T'f_n) \subset \text{supp}(|T'|(|f_n|)) \subset \overline{A_n}$ and $d(\overline{A_n}) \rightarrow 0$.

(C') \Rightarrow (A). For every $\varepsilon > 0$ there is a compact B as in (C') with the additional property that $\nu|_B$ is strictly positive and $(\chi_B T)': M(B) \rightarrow M(X)$ is defined and w^* -continuous (by Proposition 2.4). For a fixed $y \in B$ we choose sets B_n open in B with $\bigcap B_n = \{y\}$ and $d(B_n) \rightarrow 0$. If $f_n = \nu(B_n)^{-1} \chi_{B_n}$ then $f_n \cdot \nu \xrightarrow{w^*} \delta_y$ and

$$(T'f_n)\mu = (\chi_B T)'(f_n \cdot \nu) \xrightarrow{w^*} (\chi_B T)'\delta_y = \mu_y,$$

where $(\mu_y)_{y \in B}$ is the representation of $\chi_B T$. By (C'), $d(\text{supp}(T'f_n)) \rightarrow 0$ and μ_y has to be a point measure. Again, put $\sigma(y) = \text{supp} \mu_y$ and choose $a(y)$ such that $\mu_y = a(y) \cdot \delta_{\sigma(y)}$. \square

Since every bounded sequence $f_n \in L_1(\mu)$ with $f_n \xrightarrow{\mu} 0$ has a subsequence whose functions are 'almost' pairwise disjoint, the following notion may be considered a weakening of the disjointness preserving property of Riesz homomorphisms and of (C') in Proposition 5.1.

5.2 DEFINITION. Let E and F be Riesz function spaces over (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) , respectively. A linear operator $T: E \rightarrow F$ is called u - ν -continuous, if for every $L_1(X, \mu)$ -bounded sequence $f_n \in E$ with $f_n \xrightarrow{\mu} 0$ we have $Tf_n \xrightarrow{\nu} 0$.

In the same way condition (D) below may be understood as a weak form of the Maharam property and of (D) in 5.1.

5.3 THEOREM. Let E and F be as in 5.2 with $1 \in E^\times$, $1 \in F^\times$. For an order-continuous operator $T: E \rightarrow F$ the following are equivalent.

(A) T has a singular representation.

(B) If S is an integral operator with $0 \leq S \leq |T|$ then $S = 0$.

(C) For all $\varepsilon > 0$ there is a $B \in \mathcal{B}$ with $\nu(B^c) \leq \varepsilon$ such that $T\chi_B$ is u - ν -continuous.

(D) For all $\varepsilon > 0$ there is a $B \in \mathcal{B}$, $\nu(B^c) \leq \varepsilon$, such that for all $B_n \in \mathcal{B}_B$, $B_n \xrightarrow{\nu} 0$, and $a > 1$ there are $A_n \in \mathcal{A}$ with $\mu(A_n) \rightarrow 0$ and

$$a|T|(\chi_{A_n}) \geq \chi_{B_n}|T|(1) \quad \text{for all } n.$$

(E) For all $\varepsilon > 0$ there is an $A \in \mathcal{A}$, $\mu(A^c) \leq \varepsilon$, such that for a martingale $(\mathcal{F}_n, \Sigma_n)$ determined by $T\chi_A$ we have $\mathcal{F}_n(x) \xrightarrow{\nu} 0$ for μ -almost all $x \in A$.

Furthermore, $T: E \rightarrow F$ has a singular representation if and only if $T': F^\times \rightarrow E^\times$ has one.

PROOF. By 2.4 there are compact $X_n \subset X$, $Y_n \subset Y$ such that:

$$(5.2) \quad Y_n \subset Y_{n+1} \subset \cdots, \quad \nu(Y - \bigcup Y_n) = 0, \quad \nu|_{Y_n} \text{ strictly positive,}$$

$$(5.3) \quad X_n \subset X_{n+1} \subset \cdots, \quad \mu(X - \bigcup X_n) = 0, \quad \mu|_{X_n} \text{ strictly positive,}$$

$$(5.4) \quad \chi_{Y_n}T, \chi_{Y_n}|T|: C(X) \rightarrow C(Y_n), \quad \chi_{Y_n}|T|: L_\infty(X, \mu) \rightarrow L_\infty(Y_n, \nu),$$

$$(5.5) \quad (\chi_{Y_n}T)', (\chi_{Y_n}|T|)': L_1(Y_n, \nu) \rightarrow L_1(X, \mu) \text{ extend to } w^*\text{-continuous operators } M(Y_n) \rightarrow M(X) \text{ such that } (\chi_{Y_n}|T|)'(U_{M(Y_n)}) \text{ is uniformly tight,}$$

$$(5.6) \quad |T|\chi_{X_n}: L_1(X_n, \mu) \rightarrow L_1(Y, \nu).$$

By $(\mu_y)_{y \in Y}$ and $(\nu_x)_{x \in X}$ we denote the representation of T and $T': F^\times \rightarrow E^\times$ respectively.

(A) \Leftrightarrow (B) follows from 2.3 and 2.6.

Since a regular operator S is an integral operator if and only if S' is one, property (B) and therefore property (A) are self-dual.

(A) \Rightarrow (C). Since T has (A) if and only if T' has (A) we may just as well show that “ T has (A)” implies “ T' has (C)”.

We fix one of the Y_n and assume, in addition to (5.2)–(5.5), that all μ_y , $y \in Y_n$, are μ -singular. Let $f_m \in F_{Y_n}^\times$ be a sequence with $\|f_m\|_{L_1(Y_n, \nu)} \leq 1$ and $f_m \xrightarrow{\mu} 0$. It is enough to show that every subsequence of (f_m) has a subsequence (f_{m_k}) such that $T'f_{m_k} \xrightarrow{\mu} 0$. Hence we may assume that $(|f_m| \cdot \nu)$ w^* -converges to some $\lambda \in M_+(Y_n)$. By 3.4, λ must be an atomic measure of the form $\lambda = \sum \alpha_i \delta_{y_i}$ with $\sum |\alpha_i| < \infty$, $y_i \in Y_n$. From

(5.5) and (1.11) we obtain

$$(5.7) \quad |T'|(|f_m|) \cdot \mu = |T'|(|f_m|\nu) \xrightarrow{w^*} |T'|\lambda = \sum_i \alpha_i |T'|\delta_{y_i} = \sum_i \alpha_i |\mu_{y_i}|.$$

In particular, $|T'|\lambda$ is a μ -singular measure and by Lebesgue's decomposition there is for every $\delta > 0$ an open set $A \in \mathcal{A}$ with $\mu(A) \leq \delta$ and $|T'|\lambda(X - A) = 0$. Since $X - A$ is closed it follows from (5.7) that

$$\overline{\lim}_m \int_{A^c} |T'|(|f_m|) d\mu \leq |T'|\lambda(X - A) = 0,$$

i.e. $|T'f_m| \leq |T'|(|f_m|) \xrightarrow{\mu} 0$ for $m \rightarrow \infty$.

(C) \Rightarrow (E). Let $(\mathcal{F}_n, \Sigma_n)$ be a martingale determined by $T\chi_A$ with the additional property

$$(5.8) \quad \sup\{d(C) : C \text{ atom of } \Sigma_n\} \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

If $x \in A$ is contained in the atoms $C_n \in \Sigma_n$, then $f_n = \mu(C_n)^{-1} \chi_{C_n} \xrightarrow{u} 0$ and $\mathcal{F}_n(x) = T(f_n) \xrightarrow{\mu} 0$ if A is chosen as in (C).

(E) \Rightarrow (B). A martingale $(\mathcal{F}_n, \Sigma_n)$ determined by an integral operator $S: L_1(\mu) \rightarrow L_1(\nu)$ converges pointwise μ -a.e. to the kernel function $x \rightarrow k(\cdot, x)$ of S (cf. [7, §V.2]). If $0 \leq S \leq |T|$ it follows that $\mathcal{F}_n(x) \xrightarrow{\nu} 0$, i.e. $S = 0$.

In (C) \Rightarrow (E) only martingales with the additional property (5.8) are considered. Therefore we add

(A) \Rightarrow (E). If T has a singular representation then so does $|T'|$. Choose $A = X_n$ such that $\mu(A^c) \leq \varepsilon$ and $T\chi_A: L_1(A, \mu) \rightarrow L_1(Y, \nu)$. For a martingale $(\mathcal{G}_n, \Sigma_n)$ determined by $|T|\chi_A$ we have by Proposition 2.8

$$(5.9) \quad \mathcal{G}_n(x) \cdot \nu \xrightarrow{w^*} |\nu_x| \quad \mu\text{-a.e. in } A.$$

Fix an x with (5.9) and $|\nu_x| \perp \nu$. By Lebesgue's decomposition there is for every $\delta > 0$ an open set B in Y such that $|\nu_x|(Y - B) = 0$ and $\nu(B) \leq \delta$. From (5.9) we obtain

$$\overline{\lim}_n \int_{Y-B} |\mathcal{G}_n(x)| d\nu \leq |\nu_x|(Y - B) = 0.$$

Hence $\mathcal{G}_n(x) \xrightarrow{\nu} 0$ for μ -almost all $x \in A$. The same is true for a martingale $(\mathcal{F}_n, \Sigma_n)$ determined by $T\chi_A$ because $|\mathcal{F}_n(x)| \leq \mathcal{G}_n(x)$.

(A) \Rightarrow (D). Given $\varepsilon > 0$ and $a > 1$ choose sequences $\varepsilon_n > 0$ with $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$, $a_i > 1$ with $a_n < a_{n+1} < \dots < a$ and a $\delta_n > 0$ with $\delta_n \rightarrow 0$.

First we construct systems of sets $M_i^n \in \mathcal{A}$, $N_i^n \in \mathcal{B}$, $i = 1, \dots, s_n$, and $K_n \in \mathcal{B}$ such that for $n > 1$:

$$(5.10) \quad \text{The } M_i^n, i = 1, \dots, s_n, \text{ are closed in } X \text{ with } \mu(M_i^n) \leq \delta_n,$$

$$(5.11) \quad \text{The } N_i^n, i = 1, \dots, s_n, \text{ are open in } K_n \text{ and } \bigcup_{i=1}^{s_n} N_i^n = K_n,$$

(5.12) Every N_i^n is contained in some N_j^{n-1} and then M_i^n is contained in M_j^{n-1} ,

$$(5.13) \quad a_n \cdot |T|(\chi_{M_i^n}) \geq \chi_{N_i^n} |T|(1) \quad \nu\text{-a.e.},$$

(5.14) K_n is compact with $K_n \subset K_{n-1}$, $\nu(K_{n-1} - K_n) \leq \varepsilon_n$, and we have $\chi_{K_n} T \chi_{M_i^{n-1}}: C(M_j^{n-1}) \rightarrow C(K_n)$ for $i = 1, \dots, s_{n-1}$.

To start we choose a compact $K_1 \subset Y$ with $\mu(K_1^c) \leq \varepsilon_1$ such that all μ_y , $y \in K_1$, are μ -singular and put $s_1 = 1$, $M_1^1 = X$, $N_1^1 = K_1$. Assume now that all K_m , M_i^m , N_i^m , $m < n$, are already constructed. By 2.4 we can find a K_n satisfying (5.14). Choose some $1 < b_1 < b$ with $b \cdot a_{n-1} < a_n$. For a fixed $y \in K_n \cap N_i^{n-1}$ we use the abbreviations $N = N_i^{n-1} \cap K_n$, $M = M_i^{n-1}$. Since μ_y and μ are disjoint on M there is (by [39, II, Proposition 4.3]) a sequence of continuous functions h_n on M with

$$0 \leq h_j \leq 1, \quad \mu(h_j) \rightarrow 0, \quad |\mu_y|(h_j) \xrightarrow{j} |\mu_y|(M).$$

In particular, there is a j_0 such that

$$(5.15) \quad \mu(M_y) < \delta_n \quad \text{for } M_y = \left\{ h_{j_0} \geq \frac{b - b_1}{b \cdot b_1} \right\},$$

$$(5.16) \quad (|T| \chi_M)(y) = |\mu_y|(M) < b_1 |\mu_y|(h_{j_0}) = b_1 (|T| h_{j_0})(y).$$

Since $|T| \chi_M$ and $|T| h_{j_0}$ are continuous on K_n there is an open neighborhood N_y of y in K_n , such that (5.16) holds for all $z \in N_y$. From (5.16) we obtain, for ν -almost all $z \in N_y$,

$$\begin{aligned} b_1 (|T| \chi_{M_y})(z) &\geq b_1 |T|(h_{j_0})(z) - b_1 |T|(h_{j_0} \chi_{M - M_y})(z) \\ &\geq |T|(\chi_M)(z) - b_1 |T|\left(\frac{b - b_1}{bb_1} \chi_M\right)(z) = \frac{b_1}{b} |T|(\chi_M)(z). \end{aligned}$$

Since $b \cdot a_{n-1} < a_n$ and by (5.13) for M and N we get

$$(5.17) \quad a_n (|T| \chi_{M_y})(z) \geq a_{n-1} |T|(\chi_M)(z) \geq \chi_{N_y} |T|(1)(z)$$

for ν -almost all $z \in N_y \subset N$. We can find such N_y and M_y for all $y \in K_n$. Since K_n is compact there are finitely many $N_{y_1}, \dots, N_{y_{s_n}}$ that cover K_n . Put $N_i^n = N_{y_i}$ and $M_i^n = M_{y_i}$, $i = 1, \dots, s_n$. This completes the construction of N_i^n , M_i^n , K_n with (5.10) to (5.14). Finally define $B = \bigcap_n K_n$. By (5.14) we have $\nu(B^c) \leq \sum \varepsilon_n < \varepsilon$.

Let $B_k \in \mathcal{B}_B$ be a sequence with $B_k \xrightarrow{u} 0$. We may assume without loss of generality that the B_k are open in B . Indeed, for $\tilde{B}_k = \{y \in B: d(B_k, y) < 1/k\}$ we have for all subsequences (k_l)

$$\bigcap_m \bigcup_{l=m}^{\infty} B_{k_l} = \bigcap_m \bigcup_{l=m}^{\infty} \tilde{B}_{k_l}.$$

Define $A_k = \bigcup \{M_i^n: N_i^n \subset B_k\}$. From (5.11) and (5.12) it follows that $a \cdot |T|(\chi_{A_k}) \geq \chi_{B_k}(T1)$ ν -a.e. and it remains to show that $\mu(A_k) \rightarrow 0$. Otherwise there would be a subsequence k_l such that $\mu(A_{k_l}) \geq \delta > 0$. Since $B_k \xrightarrow{u} 0$ we may assume that

$C = \bigcap_m C_m$, $C_m = \overline{\bigcup_{l=m}^\infty B_{k_l}}$, is countable. Therefore, for a sequence (n_j) with $\sum \delta_{n_j} < \delta/2$, the countable set C is contained in a set of the form $N_0 = \bigcup_j N_{i_j}^{n_j}$. Observe that for $M_0 = \bigcup_j M_{i_j}^{n_j}$ we have by (5.10) $\mu(M_0) \leq \sum \delta_{n_j} < \delta/2$. On the other hand, since $N_0^c \cap \bigcap C_m = \emptyset$ and all the sets involved are compact, there is an m_0 such that $C_{m_0} \subset N_0$. But $B_{k_{m_0}} \subset C_{m_0} \subset N_0$ implies $A_{k_{m_0}} \subset M_0$ by (5.12) and the definitions of A_k . This contradicts $\mu(A_{k_{m_0}}) > \delta$ and therefore we showed that $\mu(A_k) \rightarrow 0$.

(D) \Rightarrow (B). Given $0 \leq S \leq |T|$ and $\varepsilon > 0$ there is a $B \in \mathcal{B}$, $\nu(B^c) \leq \varepsilon$, such that $|T|: L_\infty(X, \mu) \rightarrow L_\infty(B, \nu)$ has (D) and $S: L_\infty(X, \mu) \rightarrow L_\infty(B, \nu)$ is a compact integral operator (cf. [7, §V.2, Proposition 21]) with (D). Choose a sequence $B_n \in \mathcal{B}_B$ with $d(B_n) \rightarrow 0$ and $\|\chi_{B_n} S\|_\infty \geq \frac{1}{2} \|\chi_B S\|_\infty$. By (D) there are $A_n \in \mathcal{A}$, $\mu(A_n) \rightarrow 0$ with $2S(\chi_{A_n}) \geq \chi_{B_n} \cdot S1$. Since S is compact it follows that $\|S\chi_{A_n}\|_\infty \rightarrow 0$, i.e. $\chi_B S = 0$. \square

5.4. REMARK. If E and F are Köthe function spaces one might ask if one can replace in (C) the L_1 -boundedness of the sequences $f_n \in E_A$ by E -boundedness. In general the answer is no: Every continuous linear operator $T: L_p(X, \mathcal{A}, \mu) \rightarrow L_p(Y, \mathcal{B}, \nu)$, $2 < p < \infty$, would have this modified property (C). (Indeed, by Theorem 2 and Corollary 5 in [18], for a sequence $f_n \in L_p$, $2 < p < \infty$, with $\|f_n\|_{L_p} = 1$, $f_n \xrightarrow{w} 0$ the following statements are equivalent:

(i) $f_n \xrightarrow{\mu} 0$,

(ii) every subsequence of (f_n) has a subsequence equivalent to the unit vector basis of l_p ,

(iii) (f_n) has no subsequences equivalent to the unit vector basis of l_2 ,

and it is impossible that T map a basic sequence equivalent to l_p into a basic sequence equivalent to l_2 for $p > 2$.)

6. Operators with an atomic representation. Before giving characterizations analogous to 5.3 (in 6.5, 6.6) we name a class of ‘typical’ examples in $L_a(E, F)$ (compare 6.2).

6.1 DEFINITION. Let E and F be Riesz function spaces over (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) , respectively. A linear operator $T: E \rightarrow F$

(a) is called a *local Riesz homomorphism*, if there are pairwise disjoint $A_i \in \mathcal{A}$ with $\bigcup_{i \in \mathbb{N}} A_i = X$ such that $T\chi_{A_i}$ is a Riesz homomorphism for all i ,

(b) has the *local Maharam property* if there are pairwise disjoint $B_i \in \mathcal{B}$ with $\bigcup_{i \in \mathbb{N}} B_i = Y$ such that $\chi_{B_i}|T|$ has the Maharam property for all i .

Proposition 5.1 shows that these notions are dual to each other and indicates how such operators can be represented. The next theorem is due to N. J. Kalton ((i) \Leftrightarrow (ii) is given in [19, §3] with a different proof using the Kuratowski-Ryll-Nardzewski selection theorem; (ii) \Leftrightarrow (iii) is implicit in [20, §5; 19, Theorem 5.4]; the ‘locally elementary operators’ of [20] correspond to the ‘local Riesz homomorphisms’).

6.2 THEOREM. Let E and F be as in 6.1 with $1 \in E^\times$, $1 \in F^\times$. For an order-continuous operator the following are equivalent.

(i) T has an atomic representation.

- (ii) *There are Borel functions $a_n: Y \rightarrow \mathbf{R}$, $\sigma_n: Y \rightarrow X$ such that for all $n \in \mathbf{N}$ and ν -almost all $y \in Y$*

$$|a_n(y)| \geq |a_{n+1}(y)|, \quad \sum_{n=1}^{\infty} |a_n(y)| < \infty,$$

$$\sigma_n(y) \neq \sigma_m(y) \quad \text{for } m \neq n$$

and for all $f \in E$

$$Tf(y) = \sum_{n=1}^{\infty} a_n(y)f(\sigma_n(y)) \quad \nu\text{-a.e.}$$

- (iii) *There is an increasing sequence of local Riesz homomorphisms $T_n: E \rightarrow F$ such that $|T| = \sup_n T_n$.*
 (iv) *There is an increasing sequence of operators $S_n: F^\times \rightarrow E^\times$ with the local Maharam property such that $|T|' = \sup S_n$.*

As an immediate consequence of 6.2(iii), (iv) and 2.6 we obtain

6.3 COROLLARY. (a) *The band generated by all Riesz homomorphisms in $L_0(E, F)$ equals the class of all operators with an atomic representation.*

(b) *The band generated by all operators with the Maharam property in $L_0(E, F)$ equals the class of all operators whose duals have an atomic representation.*

The band generated by a single Riesz homomorphism (or operator with the Maharam property) was studied in [31].

PROOF OF 6.2. (a) \Rightarrow (b). Since (X, \mathcal{A}, μ) is a standard measure space we may assume $X = [0, 1]$. $M_a(X)$ denotes the space of all atomic measures on X and let \mathcal{B}^* be the Borel structure on $M_a(X)$ generated by the w^* -topology. Define

$$a: M_a(X) \rightarrow \mathbf{R}, \quad a(\lambda) = \max_{x \in X} |\lambda(x)|,$$

$$\phi: M_a(X) \rightarrow X, \quad \phi(\lambda) = \min\{x \in X: |\lambda(x)| = a(\lambda)\}.$$

For $A_i^n = [(i-1)/2^n, i/2^n]$, $i = 1, \dots, 2^n$, $n \in \mathbf{N}$, we have

$$a(\lambda) = \inf_n \max_{i=1}^{2^n} |\lambda(A_i^n)|.$$

This shows that a is \mathcal{B}^* -measurable. From

$$\phi^{-1}([x, 1]) = \{\lambda \in M_a(X): a(\lambda|_{[0, x)}) < a(\lambda|_{[x, 1]})\}$$

it follows that ϕ is \mathcal{B}^* -measurable since $\{\dots\}$ belongs to \mathcal{B}^* . Also $\lambda \rightarrow s(\lambda) = \text{sign } \lambda(\phi(\lambda))$ is \mathcal{B}^* -measurable (cf. [8 3.9]). Given a random measure $(\mu_y)_{y \in Y}$ representing T we put

$$a_1(y) = s(\mu_y)a(\mu_y), \quad \sigma_1(y) = \phi(\mu_y),$$

$$a_2(y) = a(\mu_y - a_1(y)\delta_{\sigma_1(y)}), \quad \sigma_2(y) = \phi(\mu_y - a_1(y)\delta_{\sigma_1(y)})$$

...

Continuing like this we obtain measurable $a_n: Y \rightarrow \mathbf{R}$, $\sigma_n: Y \rightarrow X$ such that $\mu_y = \sum_{n=1}^{\infty} a_n(y)\delta_{\sigma_n(y)}$ ν -a.e.

(b) \Rightarrow (c). By Egoroff's theorem there are $Y_n \in \mathcal{B}$ and $(m_n)_{n \in \mathbb{N}}$ with

$$(6.1) \quad Y_n \subset Y_{n+1} \subset \dots, \quad \nu\left(Y - \bigcup_n Y_n\right) = 0, \quad Y_n \text{ compact},$$

$$(6.2) \quad a_m|_{Y_n} \text{ and } \sigma_m|_{Y_n} \text{ are continuous for all } n,$$

$$(6.3) \quad \sup_{y \in Y_n} \sum_{k=m_n}^{\infty} |a_k(y)| \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

For $T_n f(y) = (\sum_{k=1}^{m_n} |a_k(y)| f(\sigma_k(y))) \chi_{Y_n}(y)$ we get $T_n(f)(y) \nearrow |T|f(y)$ ν -a.e. for all $f \in E_+$ and it remains to show that the T_n are local Riesz homomorphisms. Since all $\sigma_k|_{Y_n}$, $k = 1, \dots, m_n$, are continuous and $\sigma_k(y) \neq \sigma_l(y)$ for $k \neq l$ there is for every $x \in X$ an open neighborhood A_x such that $\sigma_1^{-1}(A_x), \dots, \sigma_{m_n}^{-1}(A_x)$ are pairwise disjoint. Define

$$\omega_x(y) = \begin{cases} \sigma_k(y) & \text{if } y \in \sigma_k^{-1}(A_x), k = 1, \dots, m_n, \\ x & \text{otherwise,} \end{cases}$$

$$a_x(y) = \begin{cases} a_k(y) & \text{if } y \in \sigma_k^{-1}(A_x), k = 1, \dots, m_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $T_n \chi_{A_x}(f)(y) = a_x(y) \cdot f(\omega_x(y))$, i.e. $T_n \chi_{A_x}$ is a Riesz homomorphism. Choose countably many $(A_{x_i})_{i \in \mathbb{N}}$ with $X = \bigcup_i A_{x_i}$ and put $A_i = A_{x_i} - \bigcup_{j=1}^{i-1} A_{x_j}$.

(c) \Rightarrow (a) is clear since $L_a(E, F)$ is a band.

(c) \Leftrightarrow (b) follows by duality from 5.1. \square

In 6.5 and 6.6 operators with an atomic representation and their duals are characterized by another weak form of the disjointness preseving property of Riesz homomorphisms, but these conditions will be stronger than the u - ν -continuity considered in 5.2, 5.3.

6.4 DEFINITION. Let E and F be Riesz function spaces over (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) , respectively. A linear operator $T: E \rightarrow F$ is called

(a) μ -continuous, if for every $L_1(X, \mu)$ -bounded sequence $f_n \in E$ with $f_n \xrightarrow{\mu} 0$ we have $Tf_n \xrightarrow{\nu} 0$,

(b) u -continuous, if for every $L_1(X, \mu)$ -bounded sequence $f_n \in E$ with $f_n \xrightarrow{u} 0$ we have $Tf_n \xrightarrow{u} 0$.

Also compare the condition 6.5(D) with 5.3(D) and 5.1(D), and condition 6.6(D') with 5.3(D) and the Maharam property.

6.5 THEOREM. Let E and F be as in 6.4 with $1 \in E^\times$, $1 \in F^\times$. For an order-continuous operator $T: E \rightarrow F$ the following are equivalent.

(A) T has an atomic representation.

(B) If Σ is a σ -subalgebra of \mathcal{A} with $\mu|_\Sigma$ diffuse and $S: E_\Sigma \rightarrow F$ is an integral operator with $0 \leq S \leq |T|_{E_\Sigma}$ then $S = 0$.

(C) For every $\epsilon > 0$ there is an $A \in \mathcal{A}$, $\mu(A^c) \leq \epsilon$, such that $T\chi_A$ is μ -continuous.

(D) For all $\varepsilon > 0$ there is a $B \in \mathcal{B}$ with $\nu(B^c) \leq \varepsilon$, such that for every $a > 1$ and every $B_n \in \mathcal{B}_B$ with $B_n \xrightarrow{u} 0$, there is a sequence $A_n \in \mathcal{A}$ with $A_n \xrightarrow{u} 0$ such that $a \cdot |T|\chi_{A_n} \geq \chi_{B_n} \cdot |T|1$.

PROOF. (A) \Rightarrow (C). It is clear that every Riesz homomorphism and every local Riesz homomorphism is μ -continuous and that the class of μ -continuous operators is closed in $L_0(L_1, L_1)$ with respect to the operator norm. But given some operator $T \in L_u(E, F)$, there is (by 6.2(iii) and Egoroff's theorem) for every $\varepsilon > 0$ an $A \in \mathcal{A}$, $\mu(A^c) \leq \varepsilon$ and a sequence of local Riesz homomorphisms T_n such that

$$\| |T|\chi_A - T_n\chi_A \|_{L_1} = \| \chi_A |T| - \chi_A T_n' \|_{L_\infty} = \| \chi_A (|T|1 - T_n'1) \|_{L_\infty} \rightarrow 0, \quad n \rightarrow \infty.$$

(C) \Rightarrow (B). For an integral operator $S: E \rightarrow F$ there is an $A \in \mathcal{A}$, $\mu(A^c) \leq \varepsilon$, such that $S\chi_A$ extends to a compact operator $S_1: L_1(\Sigma_A, \mu) \rightarrow L_1(Y, \nu)$ (cf. [7, §V.2, Proposition 21]). By [2] there is a sequence $0 \leq f_n \in L_1(\Sigma_A, \mu) \cap E_\Sigma$ with $\|f_n\| = 1$, $f_n \xrightarrow{\mu} 0$ and $\|S_1 f_n\| \rightarrow \|S_1\|_{L_1}$. If $0 \leq S \leq |T|_{E_\Sigma}$ and $|T|\chi_A$ is μ -continuous it follows that $S_1 f_n \xrightarrow{\nu} 0$. The compactness of S_1 in $L_1(\Sigma_A, \mu)$ implies $\|S_1\| = \lim \|S_1 f_n\| = 0$, i.e. $S\chi_A = 0$.

(B) \Rightarrow (A). If T is not atomic, the diffuse component $|T|^d$ of $|T|$ is not zero and by 4.2(b) there is a σ -subalgebra Σ such that $S = |T|^d|_{E_\Sigma}$ is an integral operator not equal to zero.

(A) \Rightarrow (D). By 6.2(c) there is a sequence of local Riesz homomorphisms such that $0 \leq S_n \nearrow |T|$. By Egoroff's theorem there is for every $\varepsilon > 0$, $a > a_1 > 1$, a compact $B_0 \subset Y$ with $\nu(B_0^c) \leq \varepsilon/2$ and an $S = S_n$ such that

$$(6.4) \quad a_1 S1 \geq \chi_{B_0} \cdot |T|1.$$

Let $D_i \in \mathcal{A}$, $i \in \mathbf{N}$, be a sequence of pairwise disjoint sets such that $S\chi_{D_i}$ is a Riesz homomorphism of the form

$$(6.5) \quad S\chi_{D_i}(f)(y) = a_i(y)f(\sigma_i(y)) \quad \nu\text{-a.e.}$$

Since $\sum_{i=1}^r S\chi_{D_i} \nearrow S$ there is (again by Egoroff's theorem) a compact $B \subset B_0$, $\nu(B^c) \leq \varepsilon$, and an $r \in \mathbf{N}$ such that

$$(6.6) \quad \frac{a}{a_1} \cdot \sum_{i=1}^r S\chi_{D_i} \geq \chi_B \cdot S1,$$

$$(6.7) \quad a_i|_B, \sigma_i|_B \text{ are continuous for all } i \in \mathbf{N}.$$

Given $B_m \subset B$, $B_m \xrightarrow{u} 0$, we define $A_m = \bigcup_{i=1}^r \sigma_i^{-1}(B_m)$. From (6.5), (6.6) and (6.4) it follows that

$$\begin{aligned} a|T|\chi_{A_m} &\geq a \sum_{i=1}^r S(\chi_{A_m \cap D_i}) = a \sum_{i=1}^r \chi_{\sigma_i^{-1}(A_m)} S(\chi_{D_i}) \\ &\geq a \cdot \chi_{B_m} \cdot \sum_{i=1}^r S(\chi_{D_i}) = a_1 \chi_{B_m} S(1) \geq \chi_{B_m} |T|1. \end{aligned}$$

To see that $A_m \xrightarrow{u} 0$ choose a sequence (m_l) such that $\bigcap C_n$, $C_n = \overline{\bigcup_{l=n}^{\infty} B_{m_l}}$, is countable. Since for every $i = 1, \dots, r$

$$D_i \cap \bigcap_n \overline{\bigcup_{l=n}^{\infty} A_{m_l}} \subset \bigcap_n \left[D_i \cap \bigcup_{i=1}^r \overline{\sigma_i \left(\bigcup_{l=n}^{\infty} B_{m_l} \right)} \right] \subset \bigcap_n \sigma_i(C_n) = \sigma_i \left(\bigcap C_n \right)$$

is countable (use (6.7)), it follows that $A_n \xrightarrow{u} 0$.

(D) \Rightarrow (A). If $|T|$ has property (D) then so does the diffuse component S of $|T|$. Given a B as in (D), we choose sequences $B_n \subset B$, (n_m) with

$$(6.8) \quad d(B_n) \xrightarrow{n \rightarrow \infty} 0, \quad \bigcup_{n=n_m+1}^{n_{m+1}} B_n = B.$$

By (D) there is a sequence $A_n \in \mathcal{A}$ with

$$(6.9) \quad A_n \xrightarrow{\mu} 0, \quad S\chi_{A_n} \geq \frac{1}{2}\chi_{B_n} \cdot S1.$$

From (6.8) and (6.9) we obtain

$$\chi_B \cdot S(1) = \overline{\lim}_n \chi_{B_n} S(1) \leq 2 \overline{\lim}_n S(\chi_{A_n}) = 0$$

since $A_n \xrightarrow{u} 0$ and S has a diffuse representation by 4.2(c). Hence $\chi_B S = 0$. \square

6.6 THEOREM. Let E and F be as in 6.4 with $1 \in E^\times$, $1 \in F^\times$. For an order-continuous operator $T: E \rightarrow F$ the following are equivalent.

(A) $T': F^\times \rightarrow E^\times$ has an atomic representation.

(C') For every $\varepsilon > 0$ there is an $A \in \mathcal{A}$, $\mu(A^c) \leq \varepsilon$, such that $T\chi_{A_n}$ is u -continuous.

(D') For every $\varepsilon > 0$ there is a $B \in \mathcal{B}$, $\nu(B^c) \leq \varepsilon$, such that for every $a > 1$ and every sequence $B_n \in \mathcal{B}_B$ with $\nu(B_n) \rightarrow 0$ there is a sequence $A_n \in \mathcal{A}$ with $\mu(A_n) \rightarrow 0$ and $a|T|\chi_{A_n} \geq \chi_{B_n}|T|1$.

(E') For every $\varepsilon > 0$ there is an $A \in \mathcal{A}$, $\mu(A^c) \leq \varepsilon$, such that for a martingale $(\mathcal{F}_n, \Sigma_n)$ determined by $T\chi_A$ we have $\mathcal{F}_n(x) \xrightarrow{u} 0$ μ -a.e. in A .

PROOF. (A) \Rightarrow (C'). Applying 2.4 to T' gives an $A \in \mathcal{A}$ with

$$(6.10) \quad A \text{ compact, } \mu(A^c) \leq \varepsilon, \quad \mu|_A \text{ strictly positive,}$$

$$(6.11) \quad |T|\chi_A: L_1(A, \mu) \rightarrow L_1(Y, \nu); \quad |T|\chi_A: M(A) \rightarrow M(Y) \text{ is } w^* \text{-continuous and } |T|(U_{M(A)}) \text{ is uniformly tight,}$$

$$(6.12) \quad \nu_x = |T|\chi_A(\delta_x) \text{ is atomic for all } x \in A.$$

Let (f_n) be a sequence with $\|f_n\|_{L_1(A, \mu)} \leq 1$ and $f_n \xrightarrow{u} 0$. In order to show that $Tf_n \xrightarrow{u} 0$ we may assume, in addition, that $(|f_n| \cdot \mu)_{n \in \mathbb{N}}$ w^* -converges to some $\lambda \in M(A)$. Since $|f_n| \cdot \mu \xrightarrow{u} 0$ it follows from 3.4 that λ is an atomic measure of the form $\lambda = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ where $\sum_{i=1}^{\infty} |\alpha_i| < \infty$ and $x_i \in A$. Then by (6.12),

$$|T|\lambda = \sum_{i=1}^{\infty} \alpha_i |T|(\delta_{x_i}) = \sum_{i=1}^{\infty} \alpha_i |\nu_{x_i}|$$

is an atomic measure and from $|T|(f_n) \cdot \xrightarrow{w^*} |T|\lambda$ together with 3.4 it follows that $|Tf_n| \leq |T|(|f_n|) \xrightarrow{u} 0$.

(C') \Rightarrow (E') is proved similarly to (C) \Rightarrow (E) of 5.3 (also the proof of 5.3, (A) \Rightarrow (E) can be adapted to the present situation by using 3.4).

(E') \Rightarrow (A). Let (\mathcal{F}_n, Σ) and A be as in (E'). By 2.8 there is an $A_0 \subset A$, $\mu(A - A_0) = 0$, such that $\mathcal{F}_n(x) \cdot \nu \xrightarrow{w^*} \nu_x$ for all $x \in A_0$. If λ_x is a w^* -limit of $|\mathcal{F}_n(x)| \cdot \nu$, then $|\nu_x| \leq \lambda_x$ for $x \in A_0$. Since $|\mathcal{F}_n(x)| \cdot \mu \xrightarrow{u} 0$ it follows from 3.4 that λ_x and ν_x are atomic for $x \in A_0$.

(A) \Rightarrow (D'). By 6.2(iv) there is a sequence of operators $S_n: E \rightarrow F$ with the local Maharam property such that $0 \leq S_n \nearrow |T|$. For every $\varepsilon > 0$ and $a > 1$ there is a $B_0 \subset Y$ with $\nu(B_0^c) \leq \varepsilon/2$ and an $S = S_n$ with

$$(6.13) \quad a\chi_{B_0} \cdot S(1) \geq \chi_{B_0}|T|(1).$$

By Definition 6.1 there is a $B \subset B_0$, $\nu(B^c) \leq \varepsilon$, and finitely many pairwise disjoint C_1, \dots, C_r such that $B = C_1 \cup \dots \cup C_r$ and all $\chi_{C_i}S$, $i = 1, \dots, r$, are operators with the Maharam property. By 5.1 there are ν -orthogonal random measures $(\nu_y^{(i)})_{y \in C_i}$ with Borel maps $\sigma_i: X \rightarrow C_i$ and measurable maps $b_i: X \rightarrow \mathbb{R}$ such that for all $f \in E$

$$(6.14) \quad \chi_{C_i}S(f)(y) = \int_{\sigma_i^{-1}(y)} f \cdot b_i d\nu_y^{(i)} \quad \nu\text{-a.e. in } C_i.$$

Given $B_n \subset C_i$ it follows from (6.14) that $A_n = \sigma_i^{-1}(B_n) \cap \{b_i > 0\}$ satisfies

$$(6.15) \quad \chi_{C_i}S(\chi_{A_n})(y) = \chi_{B_n}S(1)(y) \quad \nu\text{-a.e. in } C_i.$$

If $\nu(B_n) \rightarrow 0$ then $\mu(A_n) \rightarrow 0$ because

$$\int_{B_n} S(1) d\nu = \int \chi_{C_i} \cdot S(\chi_{A_n}) d\nu = \int_{A_n} S'(\chi_{C_i}) d\mu.$$

For general $B_n \subset B$ we put $A_n = \bigcup_{i=1}^r A_{n,i}$, $A_{n,i} = \sigma_i^{-1}(B_n) \cap \{b_i > 0\}$. (6.13) and (6.15) gives

$$S(\chi_{A_n}) \geq \sup_{i=1}^r S(\chi_{A_{n,i}}) \geq \chi_{B_n} \cdot S1 \geq a^{-1} \cdot \chi_{B_n}|T|(1).$$

(D') \Rightarrow (A). Let B be as in (D'). For every σ -subalgebra $\Sigma \subset \mathcal{B}_B$ one can show, as in 5.3, (D) \Rightarrow (B), that every integral operator $E \rightarrow F_\Sigma$ with $0 \leq S \leq \mathcal{E}_\Sigma(\chi_B|T)$ is zero. By duality every integral operator $S': F_\Sigma^\times \rightarrow E^\times$ with $0 \leq S' \leq (\chi_B|T)'$ is zero and $(\chi_B|T)'$ has an atomic representation. \square

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